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THE ELEMENTS

OF

PLANE GEOMETRY.

BY

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BOSTON, MASS.

NEW YORK:
A. LOVELL & CO.
1899.

BY THE SAME AUTHOR.

Academic and High School Arithmetic.

FOR PREPARATORY SCHOOLS, HIGH SCHOOLS,
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12mo. 352 pp. Price, $1.00.

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Plimpton Press
H. M. PLIMPTON & CO., PRINTERS & BINDERS,
NORWOOD, MASS., U.S.A.

PREFACE.

A QUARTER of a century ago, the study of Geometry consisted chiefly in memorizing the demonstrations of a certain number of propositions. This was sufficient to enable a student to pass creditably the entrance examination at any of the leading universities. At present, however, the case is very different; the papers given at the entrance examinations demand, on the part of the student, the ability to work original exercises. To the demands thus made by the universities are largely due the great advances that have been made in the methods of teaching Geometry, whereby it has become less and less a memory study, and more and more a study for the development of the reasoning faculty.

To accomplish this end, we find in use two very different methods. In one method the student receives the old time drill in memorizing demonstrations, and is then required to work a greater or smaller number of original exercises. This method does not seem to be conducive to economy of time for the student, as he obtains but little development of the reasoning faculty while occupied in memorizing demonstrations. In the other method referred to, the student is expected to consider everything as original work, and to act accordingly. The teacher suffers a loss of time by this method, as the student needs much personal assistance in his efforts to build up a series of demonstrations.

The author believes that a method which lies between these extremes will afford the best results in return for the amount of energy expended. Accordingly, he presents to the teaching profession this book, the results of sixteen years' experience in teaching Geometry.

At the outset, the student should understand that a demonstration is in all cases a connected series of steps, each one distinct in itself, and demanding a reason for its justification. In this book only the first three demonstrations have been given in full. This is sufficient to enable a student to see the form which a complete demonstration should take. In all other cases where it has seemed best to give an outline of the demonstration, a liberal use has been made of interrogation points. In every such case the student himself should supply the reason demanded. This practice will rapidly develop the reasoning faculty. Throughout the book an endeavor has been made to give such hints as will enable the average student to do the work without further assistance.

In addition to the regular propositions, a large number of carefully graded exercises has been inserted. This number is sufficient to enable the teacher to vary from year to year the work required from a class, and it is not expected that any one student will try to work all the exercises in the book. Attention is called to the problems of computation at the end of the different books. These problems are based on the propositions which precede them, and are designed to show the practical applications of these propositions.

While no attempt has been made to preserve the order of Euclid, yet in one respect his teaching has been followed. There seem to be no sufficient reasons for keeping theorems and problems apart, especially since the latter depend upon the former for their methods of solution. Furthermore, the student ought not to be expected to make use of construction lines in a figure before learning how to draw these lines. In this book the use of such hypothetical constructions is carefully avoided, thus following Euclid in preference to many of the modern writers. The usual modern custom of representing construction lines by dotted lines has been followed.

A special effort has been made to arrange the pages so that it will never be necessary for the student to turn the page in studying a proposition.

The author desires to express his thanks to those, both teachers and students, who have encouraged him in putting his method of teaching into book form.

CHARLES A. HOBBS.

BOSTON, July 1, 1896.

SUGGESTIONS TO TEACHERS.

THE student should be taught that the first thing in undertaking the study of a theorem is to determine exactly the hypothesis and the conclusion. Unless these two parts of a theorem are kept carefully distinct, a clear course of reasoning can never be obtained. In the process of demonstration, the student should be required to give the exact reason for every step.

It is well to conduct orally a recitation upon the advance lesson. Make sure that the pupil who is reciting understands thoroughly every statement which he makes. No error, however slight, should be allowed to pass uncorrected. A careful questioning, both upon the details of the proposition under consideration and also upon further principles suggested by the proposition, often serves to keep the attention and arouse the enthusiasm of a class.

The pupil should have much experience in written work. Conciseness, as well as accuracy, should be required. The symbols and abbreviations given on pages 8 and 9 can be used to good advantage. Many theorems can be stated in an abbreviated form; for example, Prop. 27 can be stated as follows: *Alt. int.* $\angle$*s of* $\parallel$ *lines are equal.*

Some theorems and axioms state more than one truth. When such a theorem or axiom is to be quoted as the reason for a step in a demonstration, only that part which actually applies need be given. Ax. 1 and Prop. 16 are good illustrations.

The study of Geometry is so different from any studies previously undertaken that many pupils are slow to grasp the full meaning of what they are doing. "Slow and sure" should be the aim at the beginning. The author's experience leads him to believe that the best results can be obtained by devoting to Book I. fully one third of the time which the class is to spend on Plane Geometry. After Book I. has been well mastered, the pupil ought to be able to make rapid progress.

In the study of Geometry of Space, students should be encouraged to construct models. Much may be accomplished in this direction by the use of cardboard to represent planes and knitting needles to represent straight lines. In order that students may be enabled to draw diagrams relating to the sphere on a spherical surface, every class room should be provided with a globe having a blackboard surface.

SUGGESTIONS CONCERNING EXERCISES.

It is impossible to give definite rules for working exercises in Geometry. A proof of a theorem consists in combining the hypothesis with truths previously known in such a manner as to determine a new truth. This building-up process is a process of synthesis, and is the proper manner in which to present the argument which a proof contains. Unfortunately, however, this process seldom affords a clew to the beginner who is attempting to discover a proof. The reverse process, the process of analysis, is much more helpful. In this process we first assume the truth which we wish to prove, and then deduce whatever conclusions we can until we arrive at some theorem which has previously been established. By arranging the steps thus obtained in the reverse order, the proof can be given in the regular synthetic form. Many theorems can be proved only by adding certain lines to the figure, and such lines as will be useful are likely to be discovered by the process of analysis.

A similar process is helpful in the solution of problems of construction. Draw a figure which represents the construction performed, and then apply the analytical process until some known theorem is reached which will suggest the method of construction. Auxiliary lines will often help to lead up to the proper suggestion. In making a construction, a ruler for drawing straight lines and a pair of compasses for describing arcs and circumferences are the only instruments needed in addition to pencil (or pen) and paper.

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GEOMETRY.

INTRODUCTION.

1. Space is boundless, and every material body occupies a certain limited portion of space.

A limited portion of space is called a *solid.*

NOTE. The word *solid* is used in Physics to denote a body whose particles are so bound together as to form a firmly connected whole. The geometrical use of the word is entirely distinct from this.

A solid has three dimensions — *length, breadth,* and *thickness.*

The boundaries of a solid are *surfaces.* A surface has two dimensions — *length* and *breadth.*

The boundaries of a surface are *lines.* A line has one dimension — *length.*

The extremities of a line are *points.* A point has position, but no magnitude.

When a point moves continuously, it traces a line. Likewise, a surface may be generated by a moving line, and a solid by a moving surface. Hence we may conceive of a surface as existing in space without reference to its being a boundary of a solid. In like manner we may conceive of lines and points as existing independently in space.

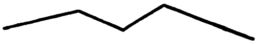
2. A *straight line,* or *right line,* is a line which has the same direction throughout its whole extent. A *curved line,* or *curve,* is a line which changes its direction at every point.

A *broken line* is a connected series of different straight lines. A *mixed line* is a connected series of lines, some of which are straight and the others curved.

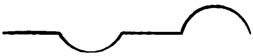
NOTE. Whenever the word *line* is used alone in this book, a straight line is meant.

A straight line.

A curved line.



A broken line.



A mixed line.

Any part of a line is called a *segment* of the line. To *produce* a line means to lengthen it at either extremity.

A *plane surface*, or *plane*, is a surface such that if any two of its points be joined by a straight line, that line lies wholly in the surface. A *curved surface* is a surface no portion of which is plane.

3. Any combination of points, lines, surfaces, or solids, with reference to form merely, is called a *figure*. A *plane figure* is a figure all of whose parts lie in the same plane. Figures containing straight lines only are called *rectilinear* figures; those containing curved lines only are called *curvilinear* figures; those containing both straight and curved lines are called *mixtilinear* figures. A *closed* figure is one which completely encloses a portion of space.

Similar figures are figures which have the same shape. *Equivalent* figures are figures which have the same size. *Equal* figures are figures which have both the same shape and size.

The equality of two figures may be tested by the method of superposition. If the figures coincide when one is applied to the other, they are equal.

4. *Geometry* is that branch of Mathematics which treats of the measurement of lines, surfaces, and solids, with their various properties and relations.

The subject-matter of Geometry contains a large number of statements called *propositions*, of which there are two classes — *theorems* and *problems*. A *theorem* is a statement of a truth which requires proof. A *problem* is a statement of a question which requires solution.

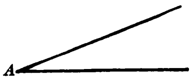
An *axiom* is a statement of a truth which is admitted to be true without proof. A *postulate* is a statement of an operation admitted to be possible. A *corollary* is a statement of a truth which can be deduced directly from the proposition which it follows. A *scholium* is an explanatory remark attached to a proposition.

Every theorem consists of two distinct parts — the *hypothesis* and the *conclusion*. The *hypothesis* is the part which is assumed to be true; in many cases it is a clause introduced by the word *if*. The *conclusion* is the part which can be proved to follow from the hypothesis.

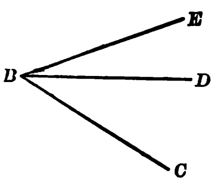
The *converse* of a theorem is a second theorem in which the hypothesis and conclusion are respectively the conclusion and hypothesis of the first.

ANGLES.

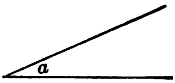
5. An *angle* is the difference in direction between two lines. The point where the lines (or lines produced) meet is called the *vertex* of the angle, and the two lines are called the *sides* of the angle.



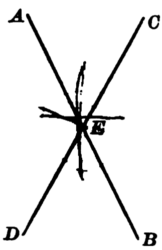
When one angle stands alone, it may be designated by the letter at its vertex, as the angle A.



When there are several angles with a common vertex, each angle is designated by three letters, of which the letter at the vertex is placed between the other two. For example, the lines BC, BD, and BE form three different angles — CBD, DBE, and CBE.



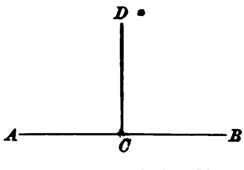
An angle is sometimes designated by a small letter or figure placed between the sides and near the vertex; as the angle a.



When two angles have the same vertex and a common side between them, they are called *adjacent* angles; as the angles BEC and CEA.

When two angles have the same vertex and the sides of one angle are the prolongations of the sides of the other, the angles are called *vertical*, or *opposite*, angles; as the angles BEC and AED.

When one straight line meets another straight line so as to form two equal adjacent angles, each angle is called a *right angle*, and the two lines are said to be *perpendicular* to each other. For example, the angles BCD and DCA are equal, and each is a right angle; the lines AB and CD are perpendicular to each other.

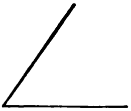


The point where a perpendicular to a line meets the line is called the *foot* of the perpendicular.

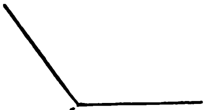
When the sides of an angle lie in the same straight line, the angle is called a *straight angle* ; as the angle BCA.

NOTE. A straight angle equals two right angles.

An angle less than a right angle is called an *acute* angle. angle larger than a right angle, but less than a straight angle, is called an *obtuse* angle. An angle larger than a straight angle is called a *reflex* angle. Acute, obtuse, and reflex angles are in general called *oblique* angles, and two intersecting lines which are not perpendicular to each other are said to be *oblique* to each other.



Acute angle.



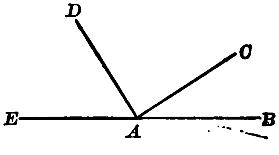
Obtuse angle.

When two straight lines lying in the same plane will never meet even if produced indefinitely, the lines are said to be *parallel* to each other.

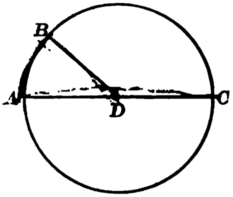
Parallel lines.

Two angles whose sum is equal to a right angle are called *complementary* angles, and either angle is said to be the *complement* of the other. Two angles whose sum is equal to a straight angle are called *supplementary* angles, and either angle is said to be the *supplement* of the other.

The size of an angle is entirely independent of the lengths of the sides. The best way to consider the size of an angle is to estimate the amount of rotation about the vertex which is necessary in order to make one side coincide with the other. Suppose the line *AB* to turn about *A* as a pivot, assuming various positions, *AC*, *AD*, *AE*, etc. The angle *BAD* is larger than the angle *BAC*, because the line *AB* must rotate more to reach the position *AD* than to reach the position *AC*.



6. In geometrical drawing it is necessary to make use of circles.



A *circle* is a portion of a plane bounded by a curved line, all the points of which are equally distant from a point within called the *centre.* The bounding line is called the *circumference,* and any part of the circumference is called an *arc;* as *AB.* An arc equal to one half the circumference is called a *semi-circumference;* as *ABC.* A straight line drawn from the centre to the circumference is called a *radius;* as *DB.* A straight line passing through the centre and having its extremities in the circumference is called a *diameter;* as *AC.*

By the definition of a circle, all the radii are equal. A diameter is the sum of two radii; hence all the diameters are equal.

POSTULATES.

7. 1. A straight line can be drawn from any one point to any other point.

2. A straight line can be produced to any length, or can be terminated at any point.

3. A circle can be described about any point as a centre with a radius of any given length.

AXIOMS.

8. *General axioms.*

1. Things which are equal to the same thing, or to equal things, are equal to each other.

2. Things which are halves of the same thing, or of equal things, are equal to each other.

3. Things which are doubles of the same thing, or of equal things, are equal to each other.

4. If equals are added to equals, the sums are equal.

5. If equals are added to unequals, the sums are unequal, and the larger sum includes the larger of the unequals.

6. If unequals are added to unequals, the sum of the larger magnitudes is larger than the sum of the smaller magnitudes.

7. If equals are subtracted from equals, the remainders are equal.

8. If equals are subtracted from unequals, the remainders are unequal, and the larger remainder is obtained from the larger of the unequals.

9. If unequals are subtracted from equals, the remainders are unequal, and the larger remainder is obtained by subtracting the smaller of the unequals.

10. If equals are multiplied by equals, the products are equal.

11. If unequals are multiplied by equals, the products are unequal, and the larger product is obtained from the larger of the unequals.

12. If equals are divided by equals, the quotients are equal.

13. If unequals are divided by equals, the quotients are unequal, and the larger quotient is obtained from the larger of the unequals.

14. Like powers or like roots of equals are equal.

15. The whole is greater than any one of its parts.

16. The whole is equal to the sum of all its parts.

Geometrical axioms.

17. Between two points only one straight line can be drawn.

18. If two straight lines have two points in common, they coincide and form but one line.

19. Two straight lines can intersect in only one point.

20. A straight line is the shortest distance between two points.

21. Two intersecting straight lines cannot both be parallel to a third straight line.

9. SYMBOLS AND ABBREVIATIONS.

+	increased by (plus).
−	diminished by (minus).
×	multiplied by.
÷	divided by.
=	is (or are) equal to.
≎	is (or are) equivalent to.
>	is (or are) greater than.
<	is (or are) less than.
∴	therefore.
∠	angle.
∠s	angles.
∥	parallel.
⊥	perpendicular.
⊥s	perpendiculars.
△	triangle.
△s	triangles.
▱	parallelogram.
▱s	parallelograms.
⊙	circle.
ⓢ	circles.
Adj.	adjacent.
Alt.	alternate.
Ax.	axiom.
Comp.	complementary.
Con.	conclusion.
Cons.	construction.
Cor.	corollary.
Corresp.	corresponding.

Def.	definition.
Ex.	exercise.
Ext.	exterior.
Hyp.	hypothesis.
Iden.	identical.
Int.	interior.
Post.	postulate.
Prop.	proposition.
Rectg.	rectangle.
Rt.	right.
St.	straight.
Sup.	supplementary.

Q.E.D., quod erat demonstrandum (which was to be proved), is placed at the end of the demonstration of a theorem.

Q.E.F., quod erat faciendum (which was to be done), is placed at the end of the solution of a problem.

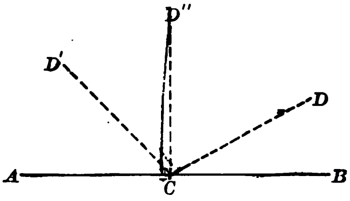
PLANE GEOMETRY.

BOOK I.

RECTILINEAR FIGURES.

Proposition 1. Theorem.

10. *At a given point in a given straight line one perpendicular to the line can be erected, and only one.*



Hypothesis. C is a point in the line AB.

Conclusion. At C one $\perp$ to AB can be erected, and only one.

Proof. Through C draw any line CD, making with AB the $\angle$s BCD and DCA, of which the $\angle BCD$ is less than the $\angle DCA$.

Now let the line CD revolve about the point C as a pivot until it takes some position as CD', where the $\angle BC$ greater than the $\angle D'CA$.

It will be possible for CD to take a position somewhere between CD and CD' such that the $\angle$s made with AB will be equal. Let CD'' be this position.

Then the $\angle BCD''$ is a rt. $\angle$, and CD'' is $\perp$ to AB.

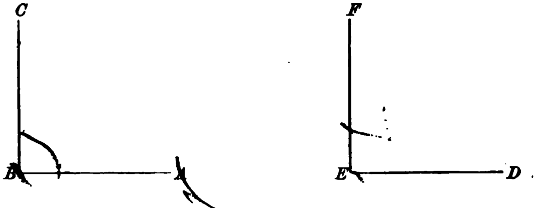
(When one st. line meets another st. line, making two equal adjacent $\angle$s, the $\angle$s are rt. $\angle$s, and the lines are $\perp$ to each other.)

If CD is moved at all from the position CD'', one $\angle$ will be enlarged at the expense of the other. Hence CD'' is the only position where the $\angle$s can be equal, and consequently CD'' is the only $\perp$ to the line AB that can be erected at the point C.

Q.E.D.

Proposition 2. Theorem.

11. *All right angles are equal.*



Hypothesis. $\angle$s ABC and DEF are rt. $\angle$s.

Conclusion. $\angle ABC = \angle DEF$.

Proof. Apply the $\angle ABC$ to the $\angle DEF$ so that the point B falls on the point E, and BA falls along ED.

Then BC will fall along EF.

(At a point in a st. line only one $\perp$ can be erected.)

$$\therefore \angle ABC = \angle DEF.$$

(Their sides coincide.)

Q.E.D.

Cor. I. *All straight angles are equal.*

Cor. II. *The complements of equal angles are equal.*

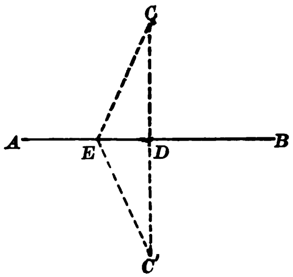
Cor. III. *The supplements of equal angles are equal.*

Scholium. The magnitude of an angle may be expressed by finding how many times it contains another angle which is taken as the unit of measure. Since right angles always have the same value, the right angle may be taken as such a unit. A more usual unit, however, is the *degree* (°), which is the ninetieth part of a right angle. A degree is divided into sixty equal parts called *minutes* (′), and a minute is divided into sixty equal parts called *seconds* (″).

For example, 18° 41′ 20″ represents an angle of 18 degrees, 41 minutes, 20 seconds.

Proposition 3. Theorem.

12. ***From a given point without a given straight line one perpendicular to the line can be drawn, and only one.***



Hypothesis. C is a point without the line AB.

Conclusion. From C one $\perp$ to AB can be drawn, and only one.

Proof. The line AB divides the plane into two parts. Let the part of the plane containing C revolve about AB as an axis until it falls on the other part of the plane, and mark the point where C falls C'.

Turn the revolved part of the plane back to its original position, and draw the straight line CC' cutting AB at D.

Draw CE, any other line from C to the line AB; also draw $C'E$.

Let the figure CED revolve about AB as an axis until C falls on C'.

Then CD will coincide with $C'D$, and CE with $C'E$.

(Between two points only one straight line can be drawn.)

$\therefore \angle CDE = \angle C'DE$, and $\angle CED = \angle C'ED$.

(Their sides coincide.)

$\angle CDE = \frac{1}{2} \angle CDC'$, and $\angle CED = \frac{1}{2} \angle CEC'$.

$\angle CDC'$ is a st. $\angle$.

(Its sides form a st. line.)

$\therefore \angle CDE$ is a rt. $\angle$.

(One half a st. $\angle$ is a rt. $\angle$.)

$\therefore CD$ is $\perp$ to AB.

(Two lines are $\perp$ to each other when they form a rt. $\angle$.)

The line CEC' is not a st. line, and $\angle CEC'$ is not a st. $\angle$.

(Between two points only one st. line can be drawn.)

$\therefore \angle CED$ is not a rt. $\angle$, but is an oblique $\angle$.

(One half of an angle which is not a st. $\angle$ is not a rt. $\angle$.)

$\therefore CE$ is oblique to AB.

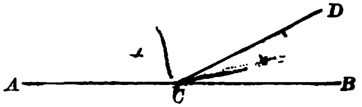
(Two lines are oblique to each other when they form an oblique $\angle$.)

$\therefore CD$ is the only $\perp$ to AB that can be drawn from C.

Q.E.D.

Proposition 4. Theorem.

13. *If one straight line meets another straight line not at its extremity, the two adjacent angles thus formed are supplementary.*



Hypothesis. The line CD meets the line AB at C.

Conclusion. $\measuredangle s\ BCD$ and DCA are supplementary.

Proof. $\angle BCD + \angle DCA = \angle BCA$. (?)

But $\angle BCA$ is a st. $\angle$. (?)

$\therefore$ $\angle BCD + \angle DCA =$ a st. $\angle$. (?)

$\therefore$ $\measuredangle s\ BCD$ and DCA are supplementary. (?)

Q.E.D.

Cor. I. *The sum of all the angles formed about a point on one side of a straight line passing through the point is equal to two right angles.*

Cor. II. *The sum of all the angles formed about a point in a plane is equal to four right angles.*

Scholium. The two angles formed when one straight line meets another straight line not at its extremity are called *supplementary adjacent* angles.

Proposition 5. Theorem.

14. *If two adjacent angles are supplementary, their exterior sides lie in the same straight line.*

Notice that this theorem is the converse of Prop. 4.

State carefully the hypothesis and the conclusion.

The steps of the proof are similar to those of Prop. 4, arranged in a different order.

Proposition 6. Theorem.

15. *If two straight lines intersect each other, the vertical angles are equal.*

Consult Prop. 4 and Prop. 2, Cor. III.

Ex. 1. The bisectors of two supplementary adjacent angles are perpendicular to each other.

Ex. 2. If the bisectors of two adjacent angles are perpendicular to each other, the angles are supplementary.

Ex. 3. The angle *BAC* is bisected by the line *AD*, and through *A* the line *EF* is drawn perpendicular to *AD*; prove that *EF* makes equal angles with *AB* and *AC*.

Ex. 4. The straight line which bisects one of two vertical angles bisects the other also.

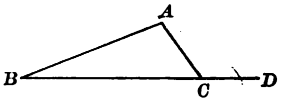
Ex. 5. The straight lines which bisect the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.

Ex. 6. If two vertical angles are bisected, the bisectors lie in the same straight line.

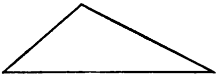
TRIANGLES.

16. A *triangle* is a portion of a plane bounded by three straight lines. The bounding lines are called the *sides* of the triangle, and their sum is called the *perimeter* of the triangle. The angles formed by the sides are called the *angles* of the triangle, and the vertices of the angles are called the *vertices* of the triangle.

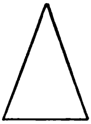
An angle formed by a side of a triangle and the prolongation of another side is called an *exterior* angle. The two interior angles not adjacent to this angle are called *opposite interior* angles. The angle *DCA* is an exterior angle, and the angles *A* and *B* are the opposite interior angles.



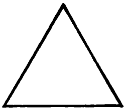
A *scalene* triangle has no two sides equal; an *isosceles* triangle has two sides equal; an *equilateral* triangle has all three sides equal.



Scalene.

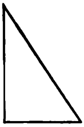


Isosceles.

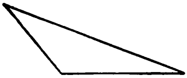


Equilateral.

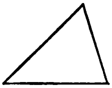
A *right* triangle is one that has a right angle; an *obtuse* triangle is one that has an obtuse angle; an *acute* triangle is one that has three acute angles. An *equiangular* triangle is one that has three equal angles.



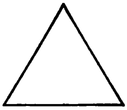
Right.



Obtuse.



Acute.



Equiangular.

In a right triangle, the side opposite the right angle is called the *hypotenuse*, and the other two sides are called the *legs*.

The side on which a triangle is supposed to stand is called the *base*. In general, any side may be taken as the base; in a right triangle, it is customary to take one of the legs as the base; in an isosceles triangle, the equal sides are called the *legs*, and the third side is taken as the base. The angle opposite the base is called the *vertical* angle, and its vertex is called the *vertex* of the triangle. The angles adjacent to the base are called *base* angles.

The perpendicular distance from the vertex to the base of a triangle (produced if necessary) is called the *altitude*.

The line drawn from the vertex to the middle point of the base of a triangle is called the *median*, or *medial line*.

NOTE. Since either side of a triangle may be taken as its base, a triangle has three altitudes and three medians.

Proposition 7. Theorem.

17. *The sum of two sides of a triangle is greater than the third side, and their difference is less than the third side.*

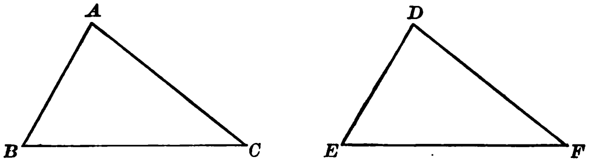
To prove the first part, consult Ax. 20.

The second part can be obtained from the first part by the use of Ax. 8.

Ex. 7. In the triangle ABC, AB is greater than AC, and D is taken on AB, so that $AD = AC$; prove that BC is greater than BD.

Proposition 8. Theorem.

18. *Two triangles are equal when a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*



Hypothesis. In the $\triangle$s ABC and DEF, $BC = EF$, $\angle B = \angle E$, and $\angle C = \angle F$.

Conclusion. $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that BC shall coincide with EF, B falling on E, and C on F.

BA will fall along ED. (?)

CA will fall along FD. (?)

$\therefore$ A, the point of intersection of BA and CA, will fall on D, the point of intersection of ED and FD.

$\therefore$ $\triangle ABC = \triangle DEF$. (?)

Q.E.D.

Scholium. Lines or angles which are similarly situated in equal or similar figures are called *homologous.* *In equal figures, the homologous parts are equal.*

Proposition 9. Theorem.

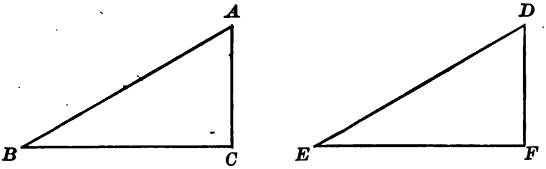
19. *Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Use the method of superposition, similar to that used in Prop. 8.

Cor. *Two right triangles are equal when the two legs of one are equal respectively to the two legs of the other.*

Proposition 10. Theorem.

20. *Two right triangles are equal when the hypotenuse and an adjacent angle of one are equal respectively to the hypotenuse and an adjacent angle of the other.*



Hypothesis. In the rt. $\triangle$s ABC and DEF, $AB = DE$, and $\angle B = \angle E$.

Conclusion. $\triangle ABC = \triangle DEF$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that AB shall coincide with DE, A falling on D, and B on E.

BC will fall along EF. (?)

AC will fall along DF. (See Prop. 3.)

[Finish as in Prop. 8.]

Ex. 8. If the line drawn from the vertex of a triangle to the middle point of the base makes equal angles with the base, the base angles are equal.

Ex. 9. *D* is a point in the bisector of the angle *BAC*. If $AB = AC$, prove that the angles *ABD* and *ACD* are equal.

Ex. 10. *D* is a point in the bisector of the angle *BAC*. From *D* the lines *DB* and *DC* are drawn, making the angles *ADB* and *ADC* equal; prove that $BD = CD$.

Ex. 11. Any point in the bisector of the vertical angle of an isosceles triangle is equidistant from the extremities of the base.

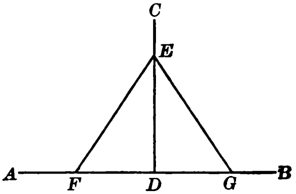
Ex. 12. The straight lines drawn from the extremities of the base of an isosceles triangle to the middle points of the opposite sides are equal.

Ex. 13. If the bisector of an angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

Ex. 14. If *D* is the middle point of the side *AB* of the triangle *ABC*, and *AE* and *BF* are perpendiculars from *A* and *B* to the line *CD*, prove that $AE = BF$.

Proposition 11. Theorem.

21. *Two oblique lines drawn from the same point in a perpendicular to a line, cutting off equal distances from the foot of the perpendicular, are equal.*



Prove that the △ *EDF* and *EDG* are equal.

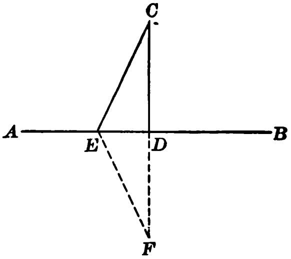
EF and *EG* are homologous sides of these △.

Cor. *Two oblique lines drawn from the same point in a perpendicular to a line, cutting off equal distances from the foot of the perpendicular, make equal angles with the base line and also with the perpendicular.*

Ex. 15. A and B are two points on the same side of the line CD; AE is drawn perpendicular to CD and is produced to F, making $EF = AE$; BF is drawn, cutting CD at G; prove that AG and BG make equal angles with CD.

Proposition 12. Theorem.

22. *The perpendicular is the shortest line that can be drawn from a point to a straight line.*



Hypothesis. CD is the ⊥ drawn from C to AB, and CE is any other line drawn from C to AB.

Conclusion. $CD < CE$.

Proof. Produce CD to F, making $DF = CD$, and draw EF.

$$CE = EF. \qquad (?)$$

$$CD + DF < CE + EF. \qquad (?)$$

$$\therefore\ 2\,CD < 2\,CE. \qquad (?)$$

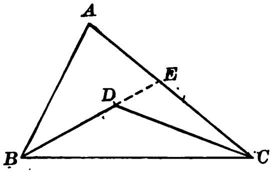
$$\therefore\ CD < CE. \qquad (?)$$

Q.E.D.

SCHOLIUM. The *distance* from a point to a line is the length of the perpendicular drawn from the point to the line.

Proposition 13. Theorem.

23. *The sum of two sides of a triangle is greater than the sum of two lines drawn from a point within the triangle to the extremities of the third side.*



Hypothesis. D is a point within the $\triangle ABC$, and DB and DC are two lines drawn from D to the extremities of BC.

Conclusion. $AB + AC > DB + DC$.

Proof. Produce BD to meet AC at E.

$$AB + AE > EB. \qquad (?)$$

$$AB + AE + EC > EB + EC. \qquad (?)$$

$$\therefore AB + AC > EB + EC. \qquad (?)$$

[Prove in a similar manner that $EB + EC > DB + DC$, and compare this result with the previous result.]

Cor. *The sum of two straight lines, drawn from a point to the extremities of a straight line, is greater than the sum of any other two straight lines similarly drawn, but included by them.*

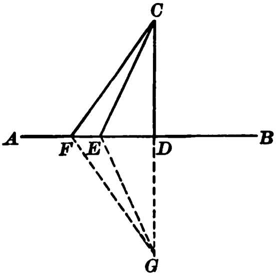
Ex. 16. The sum of the three straight lines drawn from any point within a triangle to the vertices is less than the perimeter, and is greater than one half the perimeter.

Ex. 17. If one triangle lies completely within another, the perimeter of the outer triangle is greater than the perimeter of the inner triangle.

Proposition 14. Theorem.

24. *If from the same point in a perpendicular to a line two oblique lines are drawn cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.*

Case I. *When the two oblique lines are on the same side of the perpendicular.*



Hypothesis. CD is $\perp$ to AB, and CE and CF are two oblique lines so drawn that $DF > DE$.

Conclusion. $CF > CE$.

Proof. Produce CD to G, making $DG = CD$, and draw EG and FG.

$$CE = EG, \text{ and } CF = FG. \qquad (?)$$

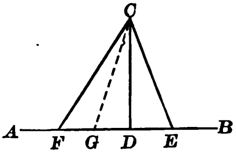
$$CF + FG > CE + EG. \qquad (?)$$

$$\therefore 2\,CF > 2\,CE. \qquad (?)$$

$$\therefore CF > CE. \qquad (?)$$

Case II. *When the two oblique lines are on opposite sides of the perpendicular.*

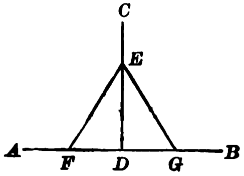
Take $DG = DE$, and draw CG. Consult Case I. and Prop. 11.



Cor. *Only two equal straight lines can be drawn from a point to a straight line.*

Proposition 15. Theorem.

25. *Two equal oblique lines drawn from the same point in a perpendicular to a line cut off equal distances from the foot of the perpendicular.*



Hypothesis. CD is $\perp$ to AB, and EF and EG are two equal oblique lines drawn from E, a point in CD, to AB.

Conclusion. $FD = DG$.

Proof. If $FD > DG$, $EF > EG$. (?)

If $FD < DG$, $EF < EG$. (?)

Each of these conclusions is contrary to the hypothesis that $EF = EG$.

$$\therefore FD = DG.$$

Q.E.D.

Scholium. The above proof is an illustration of the *indirect method* or *"reductio ad absurdum."* It consists in showing that an absurdity will result if the theorem is supposed to be otherwise than true. This method can often be used to advantage in proving the converse of a theorem that has previously been proved.

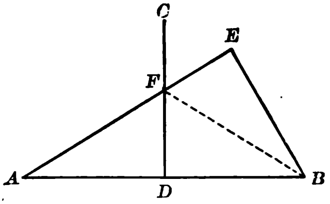
Ex. 18. State and prove the converse of Prop. 14. Use the indirect method.

Proposition 16. Theorem.

26. *Any point in the perpendicular bisector of a straight line is equally distant from the extremities of the line, and any point without the perpendicular bisector is unequally distant from the extremities of the line.*

First Part. Consult Prop. 11.

Second Part.



Hypothesis. CD is $\perp$ to AB at its middle point, and E is a point without the $\perp$.

Conclusion. EA and EB are unequal.

Proof. Since E is without the $\perp$, either EA or EB must intersect the $\perp$. Let EA intersect CD at F, and draw FB.

$$FA = FB. \qquad (?)$$

$$EF + FB > EB. \qquad (?)$$

$$\therefore\ EF + FA > EB. \qquad (?)$$

$$\therefore\ EA > EB. \qquad (?)$$

Q.E.D.

Cor. I. *Every point which is equally distant from the extremities of a straight line lies in the perpendicular bisector of the line.*

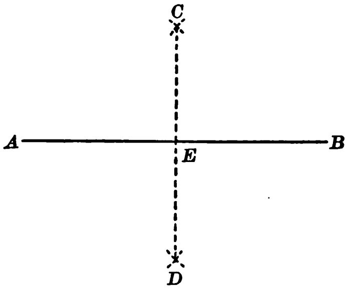
Cor. II. *Two points, each equally distant from the extremities of a straight line, determine the perpendicular bisector of the line.*

Ex. 19. If two isosceles triangles are on the same base, the straight lines joining their vertices, or that straight line produced, bisects the base at right angles.

Ex. 20. Two triangles ABC and ABD are on the same base AB and are on opposite sides of it; if $AC = AD$ and $BC = BD$, prove that CD is perpendicular to AB.

Proposition 17. Problem.

27. *To bisect a given straight line.*



To bisect the line AB.

Construction. With A as a centre and with a radius greater than $\frac{1}{2}AB$, describe two arcs. With B as a centre and with a radius equal to the former radius, describe two arcs intersecting the former arcs at C and D.

Draw CD intersecting AB at E.

Then E is the middle point of AB.

Proof. C and D are two points equally distant from A and B.

$\therefore$ CD is the $\perp$ bisector of AB. (?)

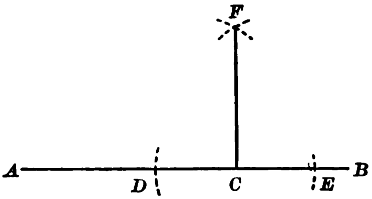
Q.E.F.

Scholium. The same method of solution may be used for the following problem:

To draw the perpendicular bisector of a given straight line.

Proposition 18. Problem.

28. *At a given point in a given straight line, to erect a perpendicular to that line.*



To erect a ⊥ to the line AB at C, a point in AB.

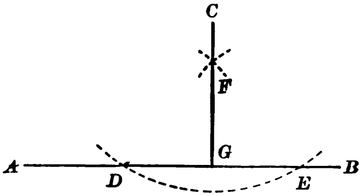
Construction. With C as a centre and with any radius, describe two arcs intersecting AB at D and E respectively.

Taking the segment DE, find the point F equidistant from D and E.

[Complete the construction and proof as in Prop. 17.]

Proposition 19. Problem.

29. *From a given point without a given straight line, to draw a perpendicular to that line.*



To draw a line ⊥ to the line AB from C, a point without AB.

Construction. With C as a centre and with a radius sufficiently great, describe an arc intersecting AB at D and E.

[Complete the construction and proof as in Prop. 17.]

Ex. 21. To find a point in a given straight line which shall be equidistant from two given points.

Ex. 22. To find a point which shall be equidistant from two given points and at a given distance from a third point.

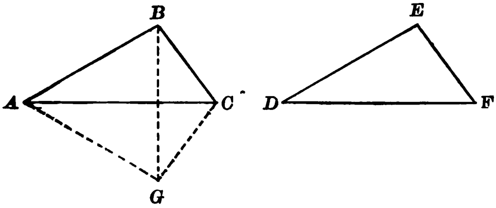
Ex. 23. To find a point which shall be at given distances from two given points.

Ex. 24. To find a point in a given straight line, such that straight lines drawn from it to two given points on the same side of the line shall make equal angles with the line.

Ex. 25. Through two given points on opposite sides of a given straight line, to draw two straight lines which shall meet in the given line, and include an angle bisected by that line.

Proposition 20. Theorem.

30. *Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.*



Hypothesis. In the $\triangle$s ABC and DEF, $AB = DE$, $AC = DF$, and $BC = EF$.

Conclusion. $\triangle ABC = \triangle DEF$.

Proof. Place the $\triangle DEF$ in the position AGC, DF coinciding with AC, and G falling on the opposite side of AC from B.

Draw BG.

AC is the $\perp$ bisector of BG. (?)

$\angle BAC = \angle GAC$. (See Prop. 11, Cor.)

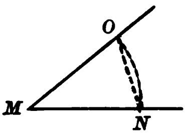
$\triangle ABC = \triangle AGC$. (?)

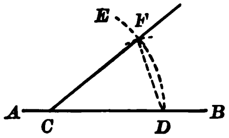
$\therefore \triangle ABC = \triangle DEF$. (?)

Q.E.D.

Proposition 21. Problem.

31. *At a given point in a given straight line, to construct an angle equal to a given angle.*





At the point *C* in the line *AB*, to construct an ∠ equal to the ∠ *M*.

Construction. With *M* as a centre and with any radius, describe an arc intersecting the sides of the ∠ *M* at *N* and *O*.

With *C* as a centre and with a radius equal to *MN*, describe the indefinite arc *DE* intersecting *AB* at *D*.

With *D* as a centre and with a radius equal to *NO*, describe an arc intersecting the arc *DE* at *F*.

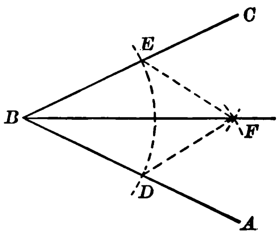
Draw *CF*.

Then the ∠ *BCF* is the required ∠.

[For proof, draw *NO* and *DF*, and consult Prop. 20.]

Proposition 22. Problem.

32. *To bisect a given angle.*



To bisect the ∠ *ABC*.

Construction. With B as a centre and with any radius, describe an arc intersecting the sides of the $\angle ABC$ at D and E.

With D and E as centres and with equal radii, each greater than $\frac{1}{2}DE$, describe arcs intersecting at F.

Draw BF.

Then BF bisects the $\angle ABC$.

[For proof, draw DF and EF, and consult Prop. 20.]

Ex. 26. To construct an angle of 45°.

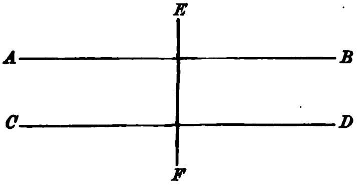
Ex. 27. To divide an angle into four equal parts.

Ex. 28. The median from the vertex to the base of an isosceles triangle is perpendicular to the base, and bisects the vertical angle.

Ex. 29. ABC and DBC are isosceles triangles on the same base; prove that the line AD bisects the angles BAC and BDC.

Proposition 23. Theorem.

33. *Two straight lines in the same plane perpendicular to the same straight line are parallel.*



Hypothesis. AB and CD are both $\perp$ to EF.

Conclusion. AB is $\parallel$ to CD.

Proof. If AB and CD are not $\parallel$, they will meet if produced sufficiently, and then there will be two $\perp$s from their point of meeting to the line EF.

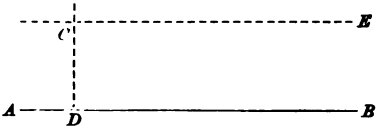
But this is impossible. (?)

$\therefore$ AB and CD cannot meet, and hence are $\parallel$.

Q.E.D.

Proposition 24. Theorem.

34. ***Through a point without a straight line one line can be drawn parallel to the given line, and only one.***



Hypothesis. *C* is a point without the line *AB*.

Conclusion. Through *C* one line can be drawn ∥ to *AB*, and only one.

Proof. Through *C* draw *CD* ⊥ to *AB*.

Through *C* draw *CE* ⊥ to *CD*.

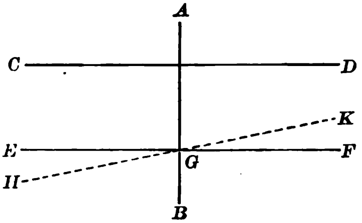
Then *CE* is ∥ to *AB*. (?)

Furthermore, *CE* is the only line that can be drawn through *C* ∥ to *AB*. (See Ax. 21.)

Q.E.D.

Proposition 25. Theorem.

35. ***If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.***



Hypothesis. *CD* is ∥ to *EF*, and *AB* is ⊥ to *CD*.

Conclusion. *AB* is ⊥ to *EF*.

Proof. Through G, the point of intersection of AB and EF, draw $HK \perp$ to AB.

Then HK is $\parallel$ to CD. (?)

But EF is $\parallel$ to CD. (?)

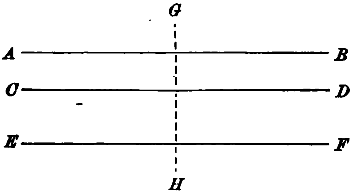
$\therefore$ HK must coincide with EF (?)

$\therefore$ AB is $\perp$ to EF. (?)

Q.E.D.

Proposition 26. Theorem.

36. ***Two straight lines parallel to the same straight line are parallel.***



Hypothesis. AB and CD are both $\parallel$ to EF.

Conclusion. AB is $\parallel$ to CD.

Proof. Through any point of EF draw $GH \perp$ to EF.

[To be completed by the student. Consult Prop. 25 and Prop. 23.]

Ex. 30. If D is the middle point of the base BC of the triangle ABC, prove that AD is greater than $\frac{1}{2}(AB + AC - BC)$.

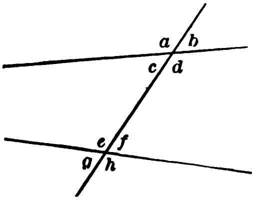
Ex. 31. ABC and DBC are two triangles on opposite sides of BC; if $AB = DC$ and $AC = DB$, prove that the angles BAD and CDA are equal.

Ex. 32. AB is the perpendicular bisector of the straight line CD; the straight line drawn through C and E, any point not in AB or CD, meets AB at F; prove that the difference between EF and DF is greater than the difference between any other two lines drawn from E and D to meet in the line AB.

37. When two or more straight lines in the same plane are cut by another line, the latter is called a *transversal.*

When a transversal crosses two lines, the eight angles thus formed are named as follows:

The four angles which lie within the two lines, *c, d, e,* and *f,* are called *interior* angles; the four angles which lie without the two lines, *a, b, g,* and *h,* are called *exterior* angles.

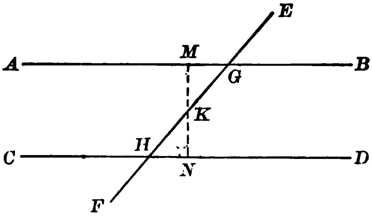


Two interior angles which are not adjacent and lie on opposite sides of the transversal are called *alternate interior* angles; as *c* and *f,* or *d* and *e.* Two exterior angles which are not adjacent and lie on opposite sides of the transversal are called *alternate exterior* angles; as *a* and *h,* or *b* and *g.*

Two angles which are not adjacent and lie on the same side of the transversal, one interior and the other exterior, are called *corresponding* angles; as *a* and *e, b* and *f, c* and *g,* or *d* and *h.*

Proposition 27. Theorem.

38. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*



Hypothesis. The ∥ lines AB and CD are cut by the transversal EF in the points G and H respectively.

Conclusion. $\angle AGH = \angle GHD$, and $\angle BGH = \angle GHC$.

Proof. Through K, the middle point of GH, draw MN $\perp$ to CD. Then MN is also $\perp$ to AB. (?)

$\triangle GKM = \triangle HKN$. (?)

$\therefore \angle AGH = \angle GHD$. (?)

$\angle BGH = \angle GHC$. (?)

Q.E.D.

SCHOLIUM. This proof cannot be used if the transversal is perpendicular to the parallel lines. The theorem is true, however, in this case; since the alternate interior angles are both right angles, they are equal.

Ex. 33. A straight line is terminated by two parallel lines, and through its middle point another straight line is drawn terminated by the two parallel lines; prove that the second straight line is bisected by the first.

Proposition 28. Theorem.

39. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

Consult Prop. 27 and Prop. 6.

Ex. 34. If two parallel lines are cut by a transversal, the alternate exterior angles are equal.

Proposition 29. Theorem.

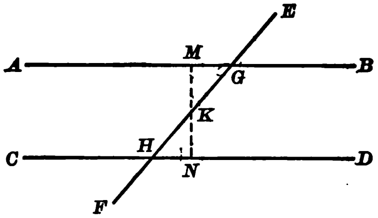
40. *If two parallel lines are cut by a transversal, the two interior angles on the same side of the transversal are supplementary.*

Consult Prop. 27 and Prop. 4.

Ex. 35. If two parallel lines are cut by a transversal, the two exterior angles on the same side of the transversal are supplementary.

Proposition 30. Theorem.

41. *If two straight lines are cut by a* ***transversal*** *so as to make the alternate interior angles* ***equal, the*** *two straight lines are parallel.*



Hypothesis. The lines AB and CD are cut by the transversal EF, so that $\angle AGH = \angle GHD$.

Conclusion. AB is $\parallel$ to CD.

Proof. Through K, the middle point of GH, draw MN $\perp$ to CD. $\triangle GKM = \triangle HKN$. (?)

$\therefore \angle GMK = \angle MNC$. (?)

$\therefore AB$ is $\perp$ to MN. (?)

$\therefore AB$ is $\parallel$ CD. (?)

Q.E.D.

Scholium. If $\angle BGH = \angle GHC$ by hypothesis, it can easily be proved that $\angle AGH = \angle GHD$, and then the proof is as given above.

If EF is perpendicular to AB and CD, the alternate interior angles are right angles, and the lines can be proved parallel by Prop. 23.

Ex. 36. If two parallel lines are cut by a transversal, the bisectors of two alternate interior angles are parallel.

Ex. 37. AB and CD are two straight lines which bisect each other; prove that AC and BD are parallel.

Proposition 31. Theorem.

42. *If two straight lines are cut by a transversal so as to make the corresponding angles equal, the two straight lines are parallel.*

Consult Prop. 6 and Prop. 30.

Ex. 38. If two straight lines are cut by a transversal so as to make the alternate exterior angles equal, the two straight lines are parallel.

Ex. 39. If two parallel lines are cut by a transversal, the bisectors of two alternate exterior angles are parallel.

Proposition 32. Theorem.

43. *If two straight lines are cut by a transversal so as to make the two interior angles on the same side of the transversal supplementary, the two straight lines are parallel.*

Consult Prop. 4 and Prop. 30.

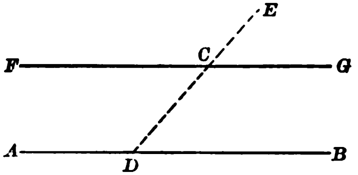
Cor. *If two straight lines are cut by a transversal so as to make the two interior angles on the same side of the transversal together less than a straight angle, the two lines are not parallel, and if produced sufficiently will meet on that side of the transversal on which the sum of the angles is less than a straight angle.*

Ex. 40. If two straight lines are cut by a transversal so as to make the two exterior angles on the same side of the transversal supplementary, the two straight lines are parallel.

Ex. 41. If two straight lines are cut by a transversal so as to make the two exterior angles on one side of the transversal together equal to the two interior angles on the same side of the transversal, the two straight lines are parallel.

Proposition 33. Problem.

44. *Through a given point without a given straight line, to draw a line parallel to the given line.*



To draw a line ∥ to AB through C, a point without AB.

Construction. From D, any point in AB, draw the line DE passing through C.

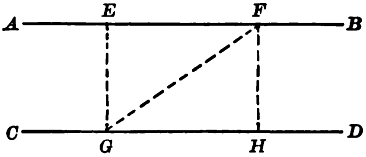
Through C draw FG, making $\angle ECG = \angle EDB$.

Then FG is ∥ to AB.

[The proof is left to the student.]

Proposition 34. Theorem.

45. *Two parallel lines are everywhere equally distant.*



Hypothesis. AB is ∥ to CD, and E and F are any two points in AB.

Conclusion. E and F are equally distant from CD.

Proof. Draw EG and FH ⊥ to CD. Also draw FG.

EG and FH are both ⊥ to AB. (?)

$\triangle EFG = \triangle FGH$. (?)

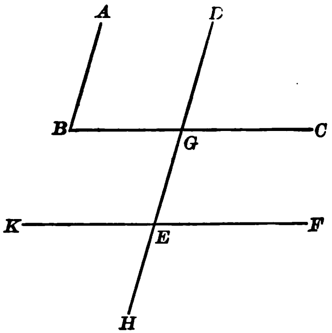
$\therefore EG = FH$. (?)

$\therefore$ E and F are equally distant from CD.

Q.E.D.

Proposition 35. Theorem.

46. *Two angles whose sides are respectively parallel to each other are either equal or supplementary.*



Hypothesis. AB is ∥ to DH, and BC is ∥ to KF.

Conclusion. $\angle ABC = \angle DEF = \angle KEH$, and $\angle ABC$ is supplementary to $\angle DEK$ and to $\angle FEH$.

Proof. Produce, if necessary, BC and DH until they intersect at G.

$\angle ABC = \angle DGC$. (?)

$\angle DGC = \angle DEF$. (?)

$\therefore \angle ABC = \angle DEF$. (?)

$\angle DEF = \angle KEH$. (?)

$\therefore \angle ABC = \angle KEH$. (?)

$\angle DEF$ is supplementary to $\angle DEK$ and to $\angle FEH$. (?)

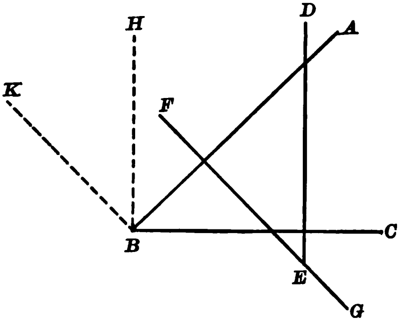
$\therefore \angle ABC$ is supplementary to $\angle DEK$ and to $\angle FEH$. (?)

Q.E.D.

Scholium. If both pairs of parallel sides extend in the same direction, or in opposite directions, from their vertices, the angles are equal; but if one pair extends in the same direction, and the other pair in opposite directions, the angles are supplementary.

Proposition 36. Theorem.

47. *Two angles whose sides are respectively perpendicular to each other are either equal or supplementary.*



Hypothesis. BA is $\perp$ to GF, and BC is $\perp$ to ED.

Conclusion. $\angle ABC = \angle FED$, and $\angle ABC$ is supplementary to $\angle DEG$.

Proof. Through B draw $BH \parallel$ to ED and $BK \parallel$ to GF.

BH is $\perp$ to BC, and BK is $\perp$ to BA. (?)

$\angle KBH = \angle ABC$. (See Prop. 2, Cor. II.)

$\angle KBH = \angle FED$. (?)

$\therefore \angle ABC = \angle FED$. (?)

$\angle FED$ is supplementary to $\angle DEG$. (?)

$\therefore \angle ABC$ is supplementary to $\angle DEG$. (?)

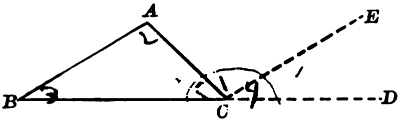
Q.E.D.

SCHOLIUM. The angles are equal if they are both acute or both obtuse; they are supplementary if one is acute and the other obtuse.

Ex. 42. Find a point B in a given straight line CD, such that if AB is drawn to B from a given point A, the angle ABC will be equal to a given angle.

Proposition 37. Theorem.

48. *The sum of the three angles of a triangle is equal to two right angles.*



Hypothesis. ABC is a $\triangle$.

Conclusion. $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s.

Proof. Produce BC to D, and draw $CE \parallel$ to BA.

Consult Prop. 4, Cor. I., Prop. 27, and Prop. 28.

Cor. I. *A triangle can have only one right angle, or one obtuse angle.*

Cor. II. *If two angles of a triangle are equal respectively to two angles of another triangle, the third angles are also equal.*

Cor. III. *If an acute angle of a right triangle is equal to an acute angle of another right triangle, the other acute angles are also equal.*

Cor. IV. *In a right triangle, the two acute angles are complementary.*

Proposition 38. Theorem.

49. *An exterior angle of a triangle is equal to the sum of the two opposite interior angles.*

Cor. *An exterior angle of a triangle is greater than either of the two opposite interior angles.*

Ex. 43. Prove Prop. 37 by drawing a line through A parallel to BC.

Ex. 44. The angle contained by the bisectors of two exterior angles of any triangle is equal to half the sum of the two adjacent interior angles.

Ex. 45. In the triangle ABC, the exterior angles at B and C are bisected by lines that meet at D; prove that the sum of the angle D and one half the angle A is equal to a right angle.

Ex. 46. In the triangle ABC, the angle B and the exterior angle at C are bisected by lines that meet at D; prove that the angle D is equal to one half the angle A.

Ex. 47. The difference of the angles at the base of a triangle is double the angle between the bisector of the vertical angle and a perpendicular drawn from the vertex to the base.

Ex. 48. The angle BAC of the triangle BAC is bisected by a line which intersects BC at D; prove that the angle ADB is greater than the angle BAD.

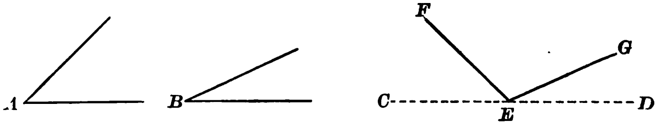
Ex. 49. If any side of a triangle is produced in both directions, the sum of the exterior angles thus formed is greater than two right angles.

Ex. 50. In the triangle ABC, the angle B is a right angle, and BC is produced to D; prove that the angle ACD is obtuse.

Ex. 51. If two straight lines are drawn from a point within a triangle to the extremities of either side, the angle included by these lines is greater than the angle included by the other two sides.

Proposition 39. Problem.

50. *Two angles of a triangle being given, to find the third angle.*



A and B are two $\angle$s of a $\triangle$; to find the third angle.

Construction. Draw the indefinite line CD.

At E, any point in CD, construct $\angle FEC = \angle A$, and $\angle DEG = \angle B$.

Then $\angle FEG$ is the required $\angle$.

[The proof is left to the student.]

Proposition 40. Theorem.

51. *Two right triangles are equal when a leg and an acute angle of one are equal respectively to a leg and the homologous acute angle of the other.*

CASE I. *When the acute angles are adjacent to the legs.* Consult Prop. 8.

CASE II. *When the acute angles are opposite the legs.* Consult Prop. 37, Cor. III. and Prop. 8.

Proposition 41. Theorem.

52. *In an isosceles triangle, the angles opposite the equal sides are equal.*

HINT. Draw a line bisecting the vertical angle.

COR. *An equilateral triangle is also equiangular.*

Proposition 42. Theorem.

53. *The bisector of the vertical angle of an isosceles triangle bisects the base and is perpendicular to the base.*

COR. *In an isosceles or an equilateral triangle, the bisector of the vertical angle, the altitude, the median, and the perpendicular bisector of the base are one and the same straight line.*

Ex. 52. Prove Prop. 41 by drawing a median to the base.

Ex. 53. The bisectors of the base angles of an isosceles triangle are equal.

Ex. 54. The altitudes upon the legs of an isosceles triangle are equal.

Ex. 55. The three altitudes of an equilateral triangle are equal.

Proposition 43. Theorem.

54. *If two angles of a triangle are equal, the sides opposite these angles are equal, and the triangle is isosceles.*

HINT. Draw the perpendicular from the vertex of the third angle to the opposite side.

COR. *An equiangular triangle is also equilateral.*

Ex. 56. Prove Prop. 43 by drawing a line bisecting the third angle.

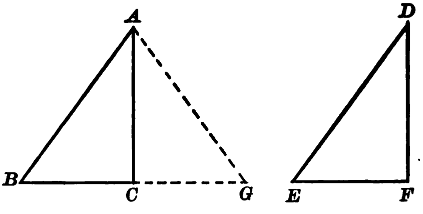
Ex. 57. If the legs of an isosceles triangle are produced, the angles on the other side of the base are equal.

Ex. 58. State and prove the converse of Ex. 57.

Ex. 59. ABC is a triangle in which $AB = AC$, and on the side BC two points D and E are taken such that $BD = CE$; prove that ADE is an isosceles triangle.

Proposition 44. Theorem.

55. *Two right triangles are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.*



Hypothesis. In the rt. $\triangle$ ABC and DEF, $AB = DE$ and $AC = DF$.

Conclusion. $\triangle ABC = \triangle DEF$.

Proof. Place the $\triangle DEF$ in the position AGC, DF coinciding with AC, and G falling on the opposite side of AC from B.

BG is a straight line. (?)

Then ABG is an isosceles $\triangle$. (?)

$\therefore \angle B = \angle G$. (?)

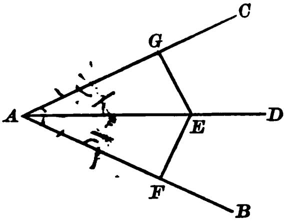
$\therefore \triangle ABC = \triangle AGC$. (?)

$\therefore \triangle ABC = \triangle DEF$. (?)

Q.E.D.

Proposition 45. Theorem.

56. *Every point in the bisector of an angle is equally distant from the sides of the angle.*



Hypothesis. E is any point in AD, the bisector of $\angle BAC$, and EF and EG are $\perp$ to AB and AC respectively.

Conclusion. $EF = EG$.

Hint. Prove that $\triangle FAE = \triangle GAE$.

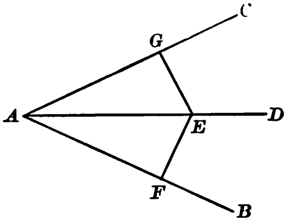
Ex. 60. If a line is drawn through the legs of an isosceles triangle parallel to the base, the triangle thus formed is isosceles.

Ex. 61. The perpendiculars dropped from the middle point of the base of an isosceles triangle upon the legs are equal.

Ex. 62. State and prove the converse of Ex. 61.

Proposition 46. Theorem.

57. *Every point within an angle, and equally distant from its sides, lies in the bisector of the angle.*



Hypothesis. E is a point within the $\angle BAC$, and EF and EG, the $\perp$ distances from E to AB and AC respectively, are equal.

Conclusion. AE bisects the $\angle BAC$.

Cor. *Any point without the bisector of an angle is unequally distant from the sides of the angle.*

Ex. 63. To find a point in a given straight line which shall be equidistant from two given intersecting straight lines.

Ex. 64. To find a point which shall be equidistant from two given points and also equidistant from two given intersecting straight lines.

Ex. 65. To find a point which shall be equidistant from two given points and also equidistant from two given parallel lines.

Ex. 66. To find a point which shall be equidistant from two given intersecting straight lines and also equidistant from two given parallel lines.

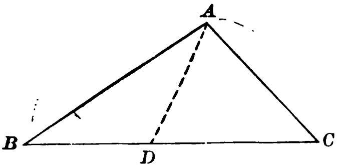
Ex. 67. To find a point in one side of a given triangle which shall be equidistant from the other two sides.

Ex. 68. If the angles B and C at the base of an isosceles triangle are bisected by the straight lines BD and CD, prove that DBC is an isosceles triangle.

Ex. 69. If a line is drawn from any point of the bisector of an angle parallel to one side, the triangle thus formed is isosceles.

Proposition 47. Theorem.

58. *If two angles of a triangle are unequal, the side opposite the greater angle is longer than the side opposite the less.*



Hypothesis. In the $\triangle ABC$, $\angle BAC > \angle B$.

Conclusion. $BC > AC$.

Proof. Draw AD, making $\angle BAD = \angle B$.

Then $BD = AD$. (?)

$AD + DC > AC$. (?)

[To be completed by the student.]

Cor. I. *In a right triangle, the hypotenuse is the longest side.*

Cor. II. *In an obtuse triangle, the side opposite the obtuse angle is the longest side.*

Ex. 70. BC, the base of an isosceles triangle ABC, is produced to any point D; prove that AD is greater than either leg.

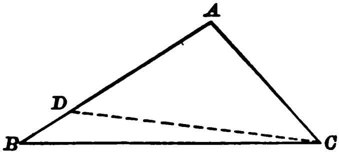
Ex. 71. In the triangle ABC, the angle A is bisected by a straight line which meets BC at D; prove that AB is greater than BD, and AC is greater than CD.

Ex. 72. Two isosceles triangles are equal when a side and the vertical angle of one are equal respectively to the homologous side and the vertical angle of the other.

Ex. 73. If on the three sides of an equilateral triangle points equally distant from the three vertices are taken in regular order and joined by straight lines, these lines form another equilateral triangle.

Proposition 48. Theorem.

59. *If two sides of a triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter.*



Hypothesis. In the triangle ABC, $AB > AC$.

Conclusion. $\angle ACB > \angle B$.

Proof. Take $AD = AC$, and draw CD.

$\angle ADC = \angle ACD$. (?)

But $\angle ADC > \angle B$. (?)

$\therefore \angle ACD > \angle B$. (?)

$\angle ACB > \angle ACD$. (?)

Much more is $\angle ACB > \angle B$.

Q.E.D.

Ex. 74. Prove Prop. 48 by the indirect method.

Ex. 75. In the triangle ABC, AB is greater than AC, and D is a point in BC; prove that AB is greater than AD.

Ex. 76. In the triangle ABC, DB and DC bisect the angles B and C respectively, and AB is greater than AC; prove that DB is greater than DC.

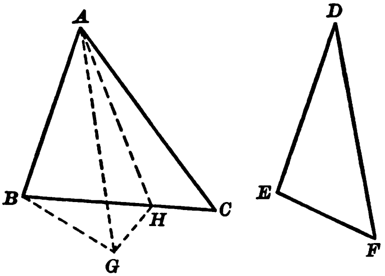
Ex. 77. If the base of an isosceles triangle is produced, the exterior angle thus formed is equal to the sum of a right angle and one half the vertical angle.

Ex. 78. The bisector of an exterior angle of an isosceles triangle, formed by producing one of the legs through the vertex, is parallel to the base.

Ex. 79. State and prove the converse of Ex. 78.

Proposition 49. Theorem.

60. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



Hypothesis. In the ⚠ ABC and DEF, $AB = DE$, $AC = DF$, and $\angle BAC > \angle D$.

Conclusion. $BC > EF$.

Proof. Place the $\triangle DEF$ in the position ABG, DE coinciding with AB. Draw AH bisecting $\angle\ GAC$, and meeting BC at H; also draw GH.

$\triangle AHG = \triangle AHC$. (?)

$\therefore\ HG = HC$. (?)

$BH + HG > BG$. (?)

$BH + HC > BG$. (?)

$\therefore\ BC > BG$. (?)

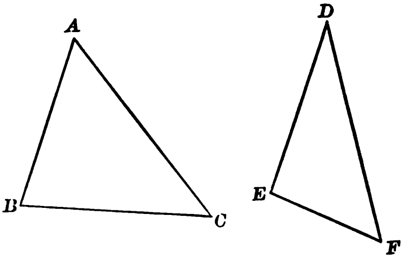
$\therefore\ BC > EF$. (?)

Q.E.D.

Ex. 80. If D is the middle point of the base BC of the triangle ABC, and the angle ADB is obtuse, prove that AB is greater than AC.

Proposition 50. Theorem.

61. ***If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the angle opposite the third side of the first is greater than the angle opposite the third side of the second.***



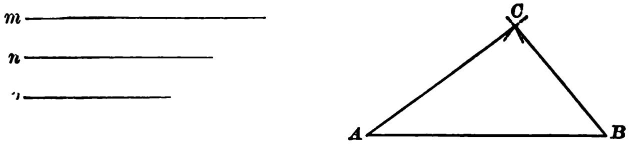
Hypothesis. In the triangles ABC and DEF, $AB = DE$, $AC = DF$, and $BC > EF$.

Conclusion. $\angle A > \angle D$.

Use the indirect method. Prove that $\angle A$ cannot be equal to $\angle D$, and $\angle A$ cannot be less than $\angle D$.

Proposition 51. Problem.

62. ***To construct a triangle whose sides shall be equal to three given straight lines.***



To construct a △ whose sides shall be equal respectively to the lines m, n, and o.

Construction. Draw $AB = m$.

With A as a centre and with a radius equal to n, describe an arc. With B as a centre and with a radius equal to o, describe an arc intersecting the former arc at C. Draw AC and BC.

Then △ ABC is the required △. Q.E.F.

Query. What relations between m, n, and o will make the problem impossible?

Scholium. The same method of solution may be used for the following problem:

To construct an equilateral triangle on a given straight line.

Ex. 81. Upon a given base to construct an isosceles triangle such that the sum of its legs shall be equal to a given straight line.

Ex. 82. To divide a right angle into three equal parts.

Proposition 52. Problem.

63. *To construct a triangle when two sides and the included angle are given.*

Proposition 53. Problem.

64. *To construct a triangle when a side and the two adjacent angles are given.*

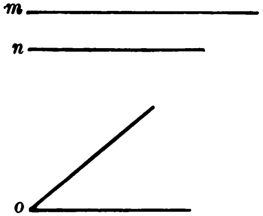
Query. What relations between the given parts will make the problem impossible?

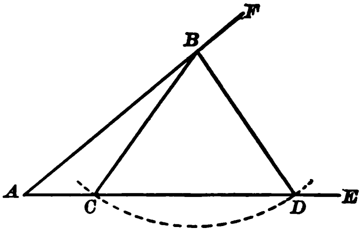
Ex. 83. To construct a right triangle when the two legs are given.

Ex. 84. To construct a right triangle when one leg and the adjacent acute angle are given.

Proposition 54. Problem.

65. *To construct a triangle when two sides and an angle opposite one of them are given.*





To construct a $\triangle$ one of whose $\angle$s shall be equal to the $\angle o$, the side opposite the $\angle$ being equal to n, and a side adjacent being equal to m.

Construction. Construct $\angle EAF = \angle o$.

On AF take $AB = m$.

With B as a centre and with a radius equal to n, describe an arc intersecting AE at C and D.

Draw BC and BD.

$\triangle ABC$ and $\triangle ABD$ both satisfy the conditions of the problem. Q.E.F.

Scholium. In the problem as given above two solutions were found *when the side opposite the given angle is less than the other given side.* Investigate this case fully, and see if the relations between the given parts can be such as to give but one solution or to make the problem impossible.

Investigate fully the cases *when the side opposite the given angle is equal to the other given side, and when the side opposite the given angle is greater than the other given side.*

The student should construct the triangles for each possible case.

Ex. 85. To construct a right triangle when the hypotenuse and one leg are given.

Ex. 86. To construct an isosceles triangle when the altitude and the base are given.

Ex. 87. To construct an isosceles triangle when a leg and a base angle are given.

Ex. 88. To construct an isosceles triangle when the altitude and the vertical angle are given.

Ex. 89. To construct an isosceles triangle when the altitude and a base angle are given.

Ex. 90. To construct an isosceles triangle when the base and the vertical angle are given.

Ex. 91. To construct an isosceles triangle when the base and the perimeter are given.

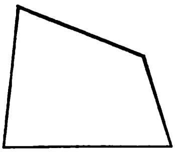
Ex. 92. To construct an isosceles triangle when the altitude and the perimeter are given.

Ex. 93. To construct an isosceles right triangle when the hypotenuse is given.

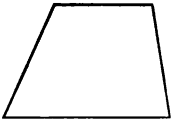
QUADRILATERALS.

66. A *quadrilateral* is a portion of a plane bounded by four straight lines. The bounding lines are called the *sides* of the quadrilateral, and their sum is called the *perimeter* of the quadrilateral. The angles formed by the adjacent sides are called the *angles* of the quadrilateral, and the vertices of the angles are called the *vertices* of the quadrilateral.

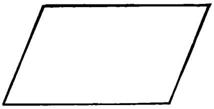
A *trapezium* is a quadrilateral which has no two sides parallel. A *trapezoid* is a quadrilateral which has two sides, and only two sides, parallel. A *parallelogram* is a quadrilateral which has its opposite sides parallel.



Trapezium.



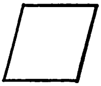
Trapezoid.



Parallelogram.

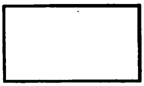
A *rhombus* is a parallelogram whose angles are oblique angles and whose sides are equal. A *rhomboid* is a parallelogram whose angles are oblique angles and whose adjacent sides are unequal.

A *rectangle* is a parallelogram whose angles are right angles. A *square* is a rectangle whose sides are equal.

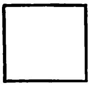


Rhombus.

Rhomboid.



Rectangle.



Square.

The side on which a parallelogram is supposed to stand and the opposite side are called respectively the *lower* and *upper* bases.

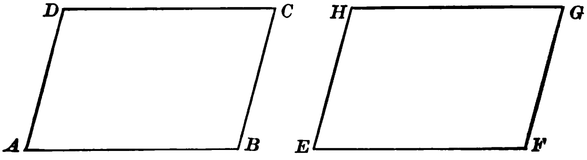
The parallel sides of a trapezoid are called the *bases*, and the other two sides are called the *legs;* the line joining the middle points of the legs is called the *median.* If the legs of a trapezoid are equal, it is called an *isosceles* trapezoid.

The perpendicular distance between the bases of a parallelogram or trapezoid is called the *altitude.*

A line joining two opposite vertices of a quadrilateral is called a *diagonal.*

Proposition 55. Theorem.

67. *Two parallelograms are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Hypothesis. In the ▱s $ABCD$ and $EFGH$, $AB = EF$, $AD = EH$, and $\angle A = \angle E$.

Conclusion. ▱ $ABCD$ = ▱ $EFGH$.

Proof. Apply the ▱ $ABCD$ to the ▱ $EFGH$ so that AB shall coincide with EF, A falling on E, and B on F.

Then AD will fall along EH. (?)
The point D will fall on the point H. (?)
BC will fall along FG. (See Prop. 24.)
DC will fall along HG. (?)
[Finish as in Prop. 8.]

Cor. *Two rectangles are equal when they have equal bases and equal altitudes.*

Proposition 56. Theorem.

68. *The opposite sides of a parallelogram are equal.*

Hint. Draw a diagonal.

Cor. I. *The segments of parallel lines included between parallel lines are equal.*

Cor. II. *A diagonal of a parallelogram divides it into two equal triangles.*

Ex. 94. To draw a straight line through a given point such that the segment intercepted between two given parallel lines shall be of given length.

Ex. 95. To draw a straight line parallel to a given straight line such that the segment intercepted between two other given intersecting straight lines shall be of given length.

Proposition 57. Theorem.

69. *The opposite angles of a parallelogram are equal.*

Consult Prop. 35.

Proposition 58. Theorem.

70. *If two sides of a quadrilateral are equal and parallel, the other two sides are equal and parallel, and the figure is a parallelogram.*

HINT. Draw a diagonal.

Proposition 59. Theorem.

71. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*

Ex. 96. The diagonals of a rhombus bisect its angles.

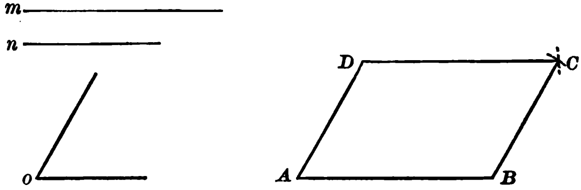
Ex. 97. The perpendiculars drawn from two opposite vertices of a parallelogram to the diagonal joining the other two vertices are equal.

Ex. 98. If in a diagonal of a parallelogram two points are taken equidistant from its extremities, and from these points lines are drawn to the opposite angles, the figure thus formed is a parallelogram.

Ex. 99. Straight lines bisecting two adjacent angles of a parallelogram are perpendicular to each other.

Proposition 60. Problem.

72. *To construct a parallelogram when two sides and the included angle are given.*



To construct a ▱, two of whose sides shall be equal respectively to *m* and *n*, the angle included by these sides being equal to *o*.

Construction. Draw $AB = m$.

At A construct $\angle A = \angle o$, and take $AD = n$.

With B as a centre, and with a radius equal to n, describe an arc. With D as a centre, and with a radius equal to m, describe an arc intersecting the former arc at C. Draw BC and DC.

Then $\square$ $ABCD$ is the required $\square$.

[The proof is left to the student.]

Scholium. By taking a right angle for the given angle, the same method of solution may be used for the following problems:

To construct a rectangle when the base and altitude are given.

To construct a square on a given straight line.

Ex. 100. To construct a rhombus when a side and an angle are given.

Ex. 101. To construct a rhombus when a side and a diagonal are given.

Ex. 102. To construct a rhomboid when two adjacent sides and a diagonal are given.

Ex. 103. The bisector of the interior angle at the vertex A of the parallelogram $ABCD$ is perpendicular to the bisector of the exterior angle at the vertex C.

Proposition 61. Theorem.

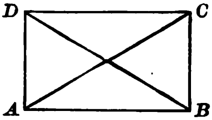
73. *If one angle of a parallelogram is a right angle, all the other angles are right angles, and the figure is a rectangle.*

Proposition 62. Theorem.

74. *The diagonals of a parallelogram bisect each other.*

Proposition 63. Theorem.

75. *The diagonals of a rectangle are equal.*



HINT. Prove that $\triangle ABD = \triangle ABC$.

Ex. 104. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Ex. 105. If the diagonals of a parallelogram are equal, the figure is a rectangle.

Ex. 106. The diagonals of a rhombus are perpendicular to each other.

Ex. 107. The diagonals of a square are perpendicular to each other, and bisect the angles of the square.

Ex. 108. If the diagonals of a quadrilateral are equal, and bisect each other at right angles, the figure is a square.

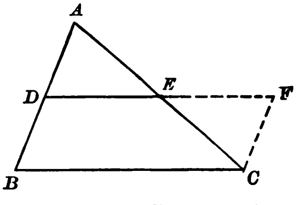
Ex. 109. To construct a square when a diagonal is given.

Ex. 110. To construct a rhombus when the two diagonals are given.

Ex. 111. To construct a rhomboid when the two diagonals and the included angle are given.

Proposition 64. Theorem.

76. *The line joining the middle points of two sides of a triangle is parallel to the third side, and is equal to one half the third side.*



Hypothesis. In the $\triangle ABC$, D and E are the middle points of AB and AC respectively.

Conclusion. DE is $\parallel$ to BC, and $DE = \frac{1}{2} BC$.

Proof. Draw $CF \parallel$ to BA, to meet DE produced at F.

$\triangle AED = \triangle CEF$. (?)

$\therefore AD = CF$. (?)

$\therefore BD = CF$. (?)

Then $BCFD$ is a ▱. (?)

$\therefore DE$ is $\parallel$ to BC. (?)

$DE = EF$, and $DE = \frac{1}{2} DF$. (?)

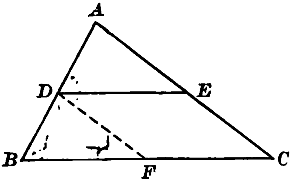
But $DF = BC$. (?)

$\therefore DE = \frac{1}{2} BC$. (?)

Q.E.D.

Proposition 65. Theorem.

77. *If a line which is parallel to one side of a triangle bisects a second side, it bisects the third side also.*



Hypothesis. In the $\triangle ABC$, DE is $\parallel$ to BC, and $AD = DB$.

Conclusion. $AE = EC$.

Proof. Draw $DF \parallel$ to AC.

$\triangle ADE = \triangle DBF$. (?)

$\therefore AE = DF$. (?)

$FCED$ is a ▱. (?)

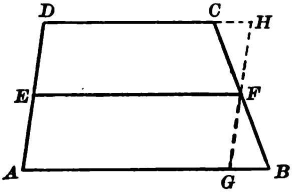
$\therefore EC = DF$. (?)

$\therefore AE = EC$. (?)

Q.E.D.

Proposition 66. Theorem.

78. *The median of a trapezoid is parallel to the parallel sides, and is equal to one half their sum.*



Hypothesis. EF is the median of the trapezoid $ABCD$.

Conclusion. EF is $\parallel$ to AB and DC, and $EF = \frac{1}{2}(AB + DC)$.

Proof. Through F draw $GH \parallel$ to AD, and produce DC to meet this line at H.

$\triangle GFB = \triangle CFH.$ (?)
$\therefore\ GF = FH.$ (?)
$AD = GH.$ (?)
$\therefore\ ED = FH.$ (?)
$EFHD$ is a ▱. (?)
$\therefore\ EF$ is $\parallel$ to DC. (?)
EF is $\parallel$ to AB also. (?)
$EF = AG = AB - GB.$ (?)
$EF = DH = DC + CH.$ (?)
But $GB = CH.$ (?)
$\therefore\ 2\,EF = AB + DC.$ (?)
$\therefore\ EF = \frac{1}{2}(AB + DC).$ (?)

Q.E.D.

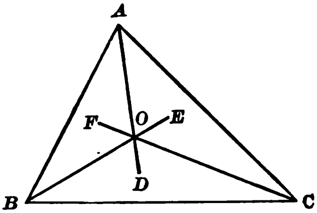
Proposition 67. Theorem.

79. *If a line which is parallel to the bases of a trapezoid bisects one leg, it bisects the other leg also.*

HINT. Draw a diagonal.

Proposition 68. Theorem.

80. *The bisectors of the three angles of a triangle intersect at a common point.*



Hypothesis. *AD, BE,* and *CF* are the bisectors of the ∠s of the △ *ABC.*

Conclusion. *AD, BE,* and *CF* intersect at a common point.

Proof. *BE* and *CF* are not ∥,
and hence intersect. (See Prop. 32, Cor.)

Let *O* be the point of intersection.

Since *O* lies in *BE*, it is equally distant from *AB* and *BC.* (?)

Since *O* lies in *CF*, it is equally distant from *AC* and *BC.* (?)

∴ *O* is equally distant from *AB* and *AC.* (?)

∴ *O* lies in *AD.* (?)

∴ *AD, BE,* and *CF* intersect at the common point *O.*

Q.E.D.

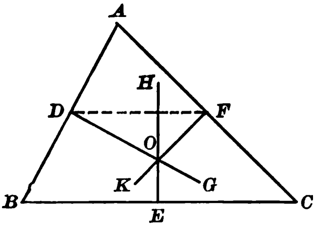
QUERY. How can the position of the point *O* be described?

Ex. 112. The bisector of the vertical angle of a triangle and the bisectors of the two exterior angles at the base intersect at a common point.

Ex. 113. In the triangle *ABC*, to find the point which is equidistant from the sides.

Proposition 69. Theorem.

81. *The perpendicular bisectors of the three sides of a triangle intersect at a common point.*



Hypothesis. DG, EH, and FK are the ⊥ bisectors of the sides of the $\triangle ABC$.

Conclusion. DG, EH, and FK intersect at a common point.

Proof. Draw DF.

DG and FK are not ∥, and hence intersect. (?)

[Consult Prop. 16 and Prop. 16, Cor. I., and finish as in Prop. 68.]

QUERY. How can the position of the point O be described?

Ex. 114. Any straight line drawn from a vertex of a triangle to the opposite side is bisected by the straight line which joins the middle points of the other two sides.

Ex. 115. Through a given point within a given angle, to draw a straight line, terminated by the sides of the angle, which shall be bisected at the point.

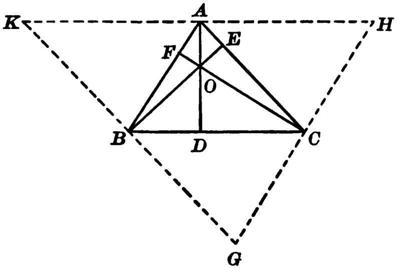
Ex. 116. The median of a trapezoid bisects both diagonals.

Ex. 117. In the trapezoid $ABCD$, E and F are the middle points of the legs AB and CD; if EF cuts AC at G and BD at H, prove that $EG = FH$.

Ex. 118. The line joining the middle points of the diagonals of a trapezoid is equal to one-half the difference of the bases.

Proposition 70. Theorem.

82. *The three perpendiculars drawn from the vertices of a triangle to the opposite sides intersect at a common point.*



Hypothesis. AD, BE, and CF are the $\perp$s drawn from the vertices of the $\triangle ABC$ to the opposite sides.

Conclusion. AD, BE, and CF intersect at a common point.

Proof. Through A draw $KH \parallel$ to BC. Likewise, through B draw $KG \parallel$ to AC, and through C draw $HG \parallel$ to AB.

AD is $\perp$ to KH. (?)

$BCAK$ and $BCHA$ are ▱s. (?)

$\therefore KA = BC$, and $AH = BC$. (?)

$\therefore KA = AH$, and A is the middle point of KH. (?)

$\therefore AD$ is the $\perp$ bisector of KH.

In like manner it may be proved that BE and CF are the $\perp$ bisectors of KG and HG respectively.

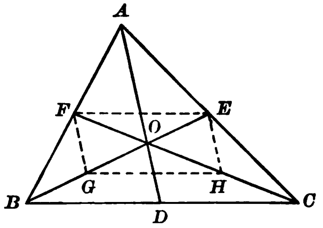
$\therefore AD$, BE, and CF intersect at a common point. (?)

Q.E.D.

Ex. 119. The straight lines joining the middle points of the sides of a triangle divide it into four equal triangles.

Proposition 71. Theorem.

83. *The three medians of a triangle intersect at a common point.*



Hypothesis. *AD, BE,* and *CF* are the medians of the △ *ABC.*

Conclusion. *AD, BE,* and *CF* intersect at a common point.

Proof. *BE* and *CF* are not ∥, and hence intersect. (?)

Let *O* be the point of intersection.

Bisect *BO* at *G,* and *CO* at *H.*

Draw *FE, FG, GH,* and *EH.*

FE and *GH* are both ∥ to *BC,* and are also both equal to $\frac{1}{2}$ *BC.* (?)

∴ *FE* and *GH* are equal and ∥. (?)

∴ *GHEF* is a ▱. (?)

∴ *FH* and *EG* bisect each other at *O.* (?)

∴ $FO = OH = HC$, and $FO = \frac{1}{3} FC$. (?)

Hence *BE* intersects *FC* at the point *O,* whose distance from *F* is equal to $\frac{1}{3}$ *FC.*

In like manner it may be proved that *AD* intersects *FC* at a point whose distance from *F* is equal to $\frac{1}{3}$ *FC*; that is, *O.*

∴ *AD, BE,* and *CF* intersect at the common point *O.*

Q.E.D.

Query. How can the position of the point *O* be described?

POLYGONS.

84. A *polygon* is a portion of a plane bounded by straight lines. The boundary lines are called the *sides* of the polygon, and their sum is called the *perimeter* of the polygon. The angles formed by the adjacent sides are called the *angles* of the polygon, and the vertices of the angles are called the *vertices* of the polygon.

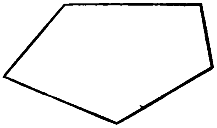
An angle formed by a side of a polygon and the prolongation of an adjacent side is called an *exterior* angle.

A line joining two vertices of a polygon which are not consecutive is called a *diagonal.*

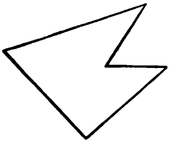
An *equilateral* polygon is one all of whose sides are equal. An *equiangular* polygon is one all of whose angles are equal. A *regular* polygon is one that is both equilateral and equiangular.

A *convex* polygon is one no side of which, if produced, will enter the space enclosed by the perimeter; each angle of such a polygon is less than a straight angle. A *concave* polygon is one of which two or more sides, if produced, will enter the space enclosed by the perimeter; an angle formed by two such sides is called a *re-entrant* angle.

NOTE. Whenever the word *polygon* is used alone in this book, a convex polygon is meant.



Convex polygon.



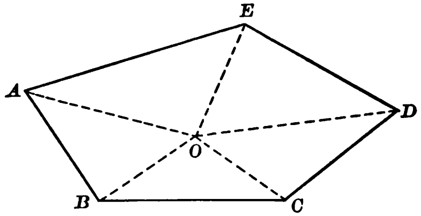
Concave polygon.

Two polygons are said to be *mutually equilateral* when the sides of one are equal respectively to the sides of the other, taken in the same order. Two polygons are said to be *mutually equiangular* when the angles of one are equal respectively to the angles of the other, taken in the same order. If two polygons are both mutually equilateral and mutually equiangular, they are equal, for one may be applied to the other so that they will coincide.

A polygon of three sides is called a *trigon* or *triangle;* one of four sides, a *tetragon* or *quadrilateral;* one of five sides, a *pentagon;* one of six sides, a *hexagon;* one of seven sides, a *heptagon;* one of eight sides, an *octagon;* one of ten sides, a *decagon;* one of twelve sides, a *dodecagon,* and so on. In general, the names of the polygons are derived from the Greek numerals which express the number of sides.

Proposition 72. Theorem.

85. *The sum of the interior angles of a polygon is equal to as many right angles as twice the number of sides less four.*



Hypothesis. *ABCDE* is a polygon of n sides.

Conclusion. The sum of the interior angles is equal to $(2n-4)$ rt. ∠s.

Proof. From any point O within the polygon draw lines to each vertex. The polygon is thus divided into n △s.

The sum of all the ∠s of these △s $= 2n$ rt. ∠s. (?)

The sum of the ∠s about O $= 4$ rt. ∠s. (?)

∴ The sum of the ∠s of the polygon $= (2n-4)$ rt. ∠s. (?)

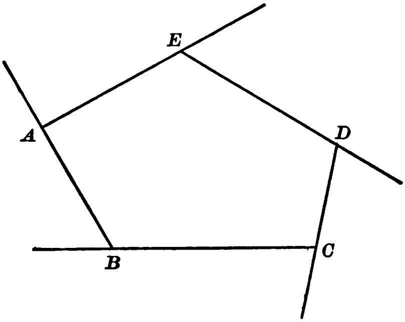
Q.E.D.

Cor. I. *The sum of the interior angles of a polygon is equal to two right angles taken as many times less two as the figure has sides.* $(n-2)\ 2$ *rt.* ∠s.

Cor. II. *Each angle of a regular polygon of n sides is equal to* $\frac{2n-4}{n}$ *rt.* ∠s.

Proposition 73. Theorem.

86. ***If the sides of a polygon are produced so as to form an exterior angle at each vertex, the sum of the exterior angles is equal to four right angles.***



Hypothesis. *ABCDE* is a polygon of n sides whose sides are produced so as to form an exterior angle at each vertex.

Conclusion. The sum of the exterior angles = 4 rt. ∠s.

Proof. The sum of all the interior and exterior ∠s

$= 2n$ rt. ∠s. (?)

The sum of the interior ∠s $= (2n-4)$ rt. ∠s. (?)

∴ The sum of the exterior ∠s = 4 rt. ∠s. (?)

Q.E.D.

LOCI.

87. It often happens in determining the position of a point that the conditions given are not sufficient to definitely locate the point, and there will be found an indefinite number of points, all of which satisfy the conditions. Furthermore, all these points will be found to lie on some definite line or group of lines. For example, if we are asked to find a point one inch from a given point, we at once find a large number of points which satisfy this condition; all of these points lie on the circumference of a circle which has the given point for its centre and has a radius one inch in length. Moreover, no points satisfy this condition except those on the circumference.

The *locus of a point* is the line or group of lines containing all the points which possess a common property, and no others.

Ex. 120. Find the locus of a point at a given distance from a given straight line.

Ex. 121. Find the locus of a point equidistant from two given parallel lines.

Ex. 122. Find the locus of a point equidistant from two given intersecting straight lines.

Ex. 123. Find the locus of a point equidistant from two given points.

Ex. 124. Find the locus of the vertex of an isosceles triangle on a given base.

Ex. 125. Find the locus of the middle point of a line drawn from a given point to a given straight line.

EXERCISES.

126. The angle formed by the bisectors of the base angles of an isosceles triangle is equal to the exterior angle at the base of the triangle.

127. If from either vertex at the base of an isosceles triangle a perpendicular is dropped upon the opposite leg, the angle between this perpendicular and the base is equal to one half the vertical angle.

128. To construct an equilateral triangle when the altitude is given.

129. To construct a triangle when the altitude and the base angles are given.

130. To construct a triangle when the altitude, the base, and a base angle are given.

131. In the triangle ABC, the angle A is double the angle B; prove that the line which bisects the angle BAC cuts BC in D so that $AD = BD$.

132. Upon a given base is constructed a triangle, one of whose base angles is double the other. The bisector of the larger base angle meets the opposite side at the point P. Find the locus of P.

133. The perpendicular erected at the middle point of one side of a triangle intersects the longer of the other two sides.

134. If BE bisects the exterior angle ABD of the triangle ABC, and AF is drawn parallel to EB meeting BC at F, the triangle ABF is isosceles.

135. If one of the legs of an isosceles triangle is produced through the vertex by its own length, the line joining the extremity of the leg produced to the nearer end of the base is perpendicular to the base.

136. If one of the angles of a parallelogram is bisected by a diagonal, the figure is either a rhombus or a square.

137. The point of intersection of the diagonals of a parallelogram bisects every straight line drawn through it and terminated by the sides of the parallelogram.

138. The sides AB and AD of a quadrilateral $ABCD$ are equal, and the diagonal AC bisects the angle BAD; prove that the sides CB and CD are equal, and that the diagonal AC bisects the angle BCD.

139. If from the vertices of a square equal distances are measured in succession on the four sides, the figure formed by joining the points thus obtained is also a square.

140. Two parallelograms are equal when two diagonals and the included angle of one are equal respectively to two diagonals and the included angle of the other.

141. Through a given point, to draw a straight line such that the perpendiculars drawn to it from two given points shall be equal. [When is the problem impossible?]

142. Through two given points, to draw straight lines which shall form with a given straight line an equilateral triangle.

143. In the isosceles triangle ABC, the equal angles B and C are bisected by DB and DC respectively; prove that AD bisects the vertical angle.

144. If the bisectors of the equal angles of an isosceles triangle meet the legs at *D* and *E*, prove that *DE* is parallel to the base of the triangle.

145. From the extremities of the base of an isosceles triangle perpendiculars are drawn to the opposite sides; prove that the angle formed by these lines is the supplement of the vertical angle.

146. If from the extremities of the base of an isosceles triangle perpendiculars are drawn to the opposite sides, the straight line joining the vertex to the point of intersection of these perpendiculars bisects the vertical angle.

147. The line joining the feet of perpendiculars let fall from the extremities of the base of an isosceles triangle upon the opposite sides is parallel to the base.

148. If the median of a triangle is equal to one half the base, the vertical angle is a right angle.

149. The straight line joining the middle point of the hypotenuse of a right triangle to the vertex of the right angle is equal to one half the hypotenuse.

150. In the right triangle *ABC*, *D* is the middle point of the hypotenuse *AB*, and *CE* is perpendicular to *AB*; prove that the angle *DCE* is equal to the difference of the angles *A* and *B*.

151. The bisector of the right angle of a right triangle bisects the angle formed by the median and the altitude drawn from the same vertex.

152. The angle between the bisectors of two consecutive angles of a quadrilateral is equal to half the sum of the two remaining angles.

153. In the quadrilateral *ABCD*, the diagonal *AC* bisects the angles *BAD* and *BCD*; prove that *AC* is perpendicular to *BD*.

154. In the quadrilateral *ABCD*, *AD* is the longest side, and *BC* is the shortest; prove that the angle *ABC* is greater than the angle *ADC*, and the angle *BCD* is greater than the angle *BAD*.

155. *AB* and *AC* are any two straight lines meeting at *A*; through any point *P*, to draw a straight line meeting them at *D* and *E* respectively, such that $AD = AE$.

156. Given two straight lines which cannot readily be produced to the point of intersection; to draw a third line which when produced would pass through the point of intersection and bisect the angle formed by the given lines.

157. *AB* and *CD* are two lines which are not parallel; through any point *P*, to draw a straight line making equal angles with the given lines.

158. *AB* and *CD* are two straight lines, and *P* is a point in *AB*; to

find a point Q in AB such that QP shall be equal to the perpendicular distance from Q to CD.

159. ABC is an isosceles triangle; to draw a straight line DE parallel to the base BC, and meeting the legs in D and E, such that $BD = DE = CE$.

160. ACB and ADB are two triangles on the same side of AB, such that $AC = BD$, and $AD = BC$, and AD and BC intersect at E; prove that the triangle AEB is isosceles.

161. On the legs AB and AC of the isosceles triangle ABC, points D and E are taken such that $AD = AE$; if BE and CD intersect at F, prove that the triangles BFC and DFE are isosceles.

162. If the two base angles of a triangle are bisected and a line parallel to the base is drawn through the point of intersection of the bisectors, the segment of this line included between the sides is equal to the sum of the segments which it cuts off from the sides.

163. If a straight line is drawn through A, one of the vertices of a square, cutting one of the opposite sides, and meeting the other produced at F, prove that AF is greater than the diagonal of the square.

164. The sum of the diagonals of a quadrilateral is less than the perimeter, but greater than one half the perimeter.

165. In the right triangle ABC, to find a point D in the hypotenuse AB, such that DB shall be equal to the perpendicular drawn from D to AC.

166. In an isosceles trapezoid, each base makes equal angles with the legs.

167. In an isosceles trapezoid, the opposite angles are supplementary.

168. The diagonals of an isosceles trapezoid are equal.

169. If the angles at the base of a trapezoid are equal, the trapezoid is isosceles.

170. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.

171. In the triangle ABC, a perpendicular is drawn from B to the bisector of the angle A, meeting it at D, and BD is produced to meet AC at E; prove that $BD = DE$.

172. In the triangle ABC, AD is the bisector of the angle A, and on BA produced, E is taken so that $AE = AC$; prove that EC is parallel to AD.

173. If a straight line is drawn bisecting one angle of a triangle to meet the opposite side, the straight lines drawn from the point of intersection parallel to the other sides, and terminated by them, will be equal.

174. The bisectors of the angles which a transversal makes with one of two parallel lines meet the other parallel at equal distances from the transversal.

175. In the triangle ABC, the side BC is produced to D, and the bisector of the angle ACB meets AB at E; through E a line is drawn parallel to BD, intersecting AC at F and the bisector of the angle ACD at G; prove that $EF = FG$.

176. If two parallel lines are cut by a transversal, the bisectors of the interior angles form a rectangle.

177. The bisectors of the interior angles of a trapezoid form a quadrilateral, two of whose angles are right angles.

178. The bisectors of the interior angles of a rhomboid form a rectangle.

179. In the quadrilateral $ABCD$, the opposite sides AD and BC are equal, and also the diagonals AC and BD are equal; if AC and BD intersect at E, prove that the triangles AEB and DEC are isosceles.

180. If the sides of a pentagon are produced, the sum of the angles formed at the points of intersection is equal to two right angles.

181. If one of the acute angles of a right triangle is double the other, the hypotenuse is double the shorter leg.

182. If one of the angles of a triangle is double the other, the triangle can be divided into two isosceles triangles by a line drawn through the vertex of the third angle.

183. The bisectors of the angles of the triangle ABC meet at O, and OD is drawn perpendicular to AB; prove that AD is equal to the difference between half the perimeter and BC.

184. In the triangle ABC, AD is drawn perpendicular to BC, and E, F, and G are the middle points of AB, AC, and BC respectively; prove that the angles BAC, EDF, and EGF are equal.

185. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other two angles are perpendicular to each other.

186. If two opposite angles of a quadrilateral are right angles, the bisectors of the other two angles are parallel.

187. AB is a straight line bisected at C, and AD and BE are perpendiculars drawn from A and B to any other straight line; prove that $CD = CE$.

188. A, B, and C are three points in a straight line, such that $AB = BC$; prove that the sum of the perpendiculars drawn from A

and C to any straight line which does not pass between A and C is double the perpendicular drawn from B to the same line.

189. If from the vertices of a parallelogram perpendiculars are drawn to any straight line, the sum of the two perpendiculars drawn from two opposite vertices is equal to the sum of the other two.

190. A line drawn perpendicular to the base of an equilateral triangle forms with the other two sides an isosceles triangle whose vertical angle is four times as large as each of the base angles.

191. The vertical angle of an isosceles triangle is four times as large as each of the base angles; prove that a line drawn perpendicular to the base forms with the legs an equilateral triangle.

192. If a straight line DE is drawn perpendicular to the base AC of the isosceles triangle ABC, cutting BC at F and AB produced at E, the triangle BEF is isosceles.

193. AD is the bisector of the angle BAC, and $AB = AC$; from B and C lines are drawn perpendicular to AB and AC respectively; prove that they intersect AD at the same point.

194. If ABC is an equilateral triangle, and if BD and CD bisect the angles B and C respectively, the lines DE and DF, drawn parallel to AB and AC respectively, trisect BC.

195. If from any point in the base of an isosceles triangle lines are drawn parallel to the legs, the perimeter of the parallelogram thus formed is constant, and is equal to the sum of the legs of the triangle.

196. The lines joining the middle points of the sides of a square, taken in order, enclose a square.

197. The lines joining the middle points of the sides of a rhombus, taken in order, enclose a rectangle.

198. The lines joining the middle points of the sides of any quadrilateral, taken in order, enclose a parallelogram. [HINT. Draw a diagonal.]

199. The two lines joining the middle points of the opposite sides of a quadrilateral bisect each other.

200. If from the vertices of a parallelogram equal distances are measured in succession on the four sides, the figure formed by joining the points thus obtained is also a parallelogram.

201. In the parallelogram $ABCD$, the points E, F, G, and H are taken on the sides AB, AD, BC, and CD respectively, such that $AE = AF = CG = CH$; prove that $EFHG$ is a parallelogram.

202. In the equilateral triangle ABC, BD is drawn perpendicular to AC, and DE is drawn perpendicular to BC; prove that $EC = \frac{1}{4}BC$.

203. In the triangle ABC, the bisector of the angle C meets AB at D, and through D a line is drawn parallel to AC cutting BC at E and the bisector of the exterior angle at C at F; prove that $DE = EF$.

204. AB and CD are two straight lines intersecting at E; the straight lines AC and BD are drawn, forming the two triangles ACE and BDE; the angles ACE and DBE are bisected by the straight lines CF and BF, meeting at F; prove that the angle CFB is equal to one half the sum of the angles EAC and EDB.

205. In the triangle ABC, the angle B is greater than the angle C, and BD is drawn to AC, making $AD = AB$; prove that the angle ADB is equal to one half the sum of the angles ABC and ACB, and the angle CBD is equal to one half the difference of the same two angles.

206. An angle of a triangle is acute, right, or obtuse, according as the median drawn from its vertex is greater than, equal to, or less than one half the opposite side.

207. In the triangle ABC, the side BC is bisected at D, and AB at E; AD is produced to F, making $DF = AD$, and CE is produced to G, making $EG = CE$; prove that the straight line FG passes through B.

208. In the parallelogram $ABCD$, E and F are the middle points of AD and BC respectively; prove that the figure $AFCE$ is a parallelogram.

209. In the parallelogram $ABCD$, E and F are the middle points of AD and BC respectively; prove that AF and CE trisect the diagonal BD.

210. If from either vertex of a parallelogram straight lines are drawn to the middle points of the two opposite sides, these lines trisect a diagonal of the parallelogram.

211. To construct a right triangle when the hypotenuse and the sum of the legs are given.

212. To construct a right triangle when the hypotenuse and the difference of the legs are given.

213. To construct a triangle when the vertical angle, the base, and the sum of the remaining sides are given.

214. To construct a triangle when the vertical angle, the base, and the difference of the remaining sides are given.

215. In the triangles ABC and DEF, AB is equal to DE, the angle ABC is equal to the angle DEF, and BC is greater than EF; prove that the angle ACB is less than the angle DFE.

216. From the extremities of the base of a triangle perpendiculars are drawn to the opposite sides; prove that the straight lines drawn from the middle point of the base to the feet of the perpendiculars are equal.

217. From the extremities of the base of a triangle perpendiculars are drawn to the opposite sides; prove that the straight line joining the feet of the perpendiculars is bisected by a perpendicular drawn to it from the middle point of the base.

218. Upon the sides of the triangle ABC, the equilateral triangles DAB, EAC, and FBC are constructed, all external to the original triangle; prove that $AF = BE = CD$.

219. If from the diagonal BD of a square $ABCD$, BE is cut off equal to BC, and EF is drawn perpendicular to BD, meeting DC at F, prove that $DE = EF = FC$.

220. If two medians of a triangle are equal, the triangle is isosceles.

221. The median drawn to any side of a triangle is less than one half the sum of the other two sides. [HINT. Produce the median beyond the base by its own length.]

222. In any triangle, the sum of the medians is less than the perimeter, and is greater than three quarters of the perimeter.

223. If the median of a triangle bisects the vertical angle, the triangle is isosceles.

224. The sum of the perpendiculars dropped from any point in the base of an isosceles triangle upon the legs is constant, and is equal to the altitude upon one of the legs.

225. The sum of the perpendiculars dropped from any point within an equilateral triangle upon the three sides is constant, and is equal to the altitude.

226. To construct a triangle when the perimeter and the base angles are given.

227. To construct a triangle when a base angle, the base, and the sum of the other two sides are given.

228. To construct a triangle when a base angle, the base, and the difference of the other two sides are given.

PROBLEMS OF COMPUTATION.

1. Find the number of degrees in an angle which is equal to one half of its complement.

2. Find the number of degrees in an angle which is equal to one third of its supplement.

3. Five lines meet at a point, and four of the angles thus formed are 42°, 55°, 88°, and 123°; find the number of degrees in the fifth angle.

4. Find the number of degrees in the angle formed by the bisectors of two complementary adjacent angles.

5. Find the number of degrees in each angle of an equilateral triangle.

6. Find the number of degrees in each acute angle of an isosceles right triangle.

7. One angle of a triangle is 45°; find the number of degrees in each of the other angles if one of them is twice as large as the other.

8. In the triangle *ABC*, the angle *B* is twice as large as the angle *A*, and the angle *C* is twice as large as the angle *B*; find the number of degrees in each angle.

9. In the triangle *ABC*, the angles *A* and *B* are 62° and 104° respectively; find the number of degrees in the exterior angle at *C*.

10. The exterior angles at *A* and *B* of the triangle *ABC* are 126° and 64° respectively; find the number of degrees in the angle *C*.

11. The vertical angle of an isosceles triangle is 72°; find the number of degrees in the exterior angle formed by producing the base.

12. An exterior angle of an isosceles triangle formed by producing the base contains 132°; find the number of degrees in the vertical angle.

13. How many degrees are there in each angle of an isosceles triangle if the vertical angle is one half as large as each of the base angles?

14. If the acute angles of a right triangle are 40° and 50°, find the number of degrees in the angle formed by their bisectors.

15. The vertical angle of an isosceles triangle is 80°; find the number of degrees in the angle formed by the bisectors of the base angles.

16. In an isosceles right triangle, the length of the altitude upon the hypotenuse is 15 inches; find the length of the hypotenuse.

17. In the isosceles triangle *ABC*, the equal angles *A* and *B* are each equal to 30°, and *AC* is 20 inches in length; find the length of the perpendicular drawn from *C* to *AB*.

18. Two sides of a triangle are 8 inches and 14 inches; between what limits is the length of the third side?

19. One angle of a parallelogram is 72°; find the number of degrees in each of the other angles.

20. The sum of two opposite angles of a parallelogram is 214°; find the number of degrees in each angle.

21. The difference of two consecutive angles of a parallelogram is 30°; find the number of degrees in each angle.

22. The lengths of the bases of a trapezoid are 8 inches and 14 inches; find the length of the median.

23. Find the sum of the angles of an octagon.

24. Find the number of degrees in each angle of a regular decagon.

25. If one side of a regular pentagon is produced, how many degrees will the exterior angle contain?

26. The sum of the angles of a polygon is equal to 32 right angles; how many sides has the polygon?

27. Each exterior angle of a regular polygon is equal to an angle of an equilateral triangle; how many sides has the polygon?

28. The sum of the interior angles of a polygon is equal to the sum of the exterior angles; how many sides has the polygon?

29. The sum of the interior angles of a polygon is equal to four times the sum of the exterior angles; how many sides has the polygon?

30. The interior angle of a regular polygon exceeds the exterior angle by 120°; how many sides has the polygon?

BOOK II.

PROPORTION. THEORY OF LIMITS.

RATIO.

88. The relative magnitude of two quantities of the same kind is called their *ratio*, and it is determined by dividing the first by the second. The division may be expressed in fractional form, or by placing a colon between the two quantities. For example, the ratio of a to b may be written $\frac{a}{b}$, or $a : b$.

The two quantities whose values are compared are called the *terms* of the ratio. The first term is called the *antecedent*, and the second term is called the *consequent.*

A ratio is called a ratio of *equality*, of *greater inequality*, or of *less inequality*, according as the antecedent is equal to, greater than, or less than the consequent.

If the terms of a ratio are interchanged, the resulting ratio is called the *inverse* of the given ratio.

MEASUREMENT.

89. A quantity is measured by finding the ratio of the quantity to another quantity of the same kind, called the *unit of measure*, and the numerical quotient is called the *numerical measure* of the quantity.

A line is measured by finding the ratio of the length of the line to the length of another line, called the *unit of length*, or *linear unit.* For example, a line which is four times as long as the linear unit called an inch is said to have a length of four inches.

When a quantity is contained in a larger quantity an integral number of times, the second is said to be a *multiple* of the first. *Equimultiples* of two or more quantities are the products obtained by multiplying these quantities by the same number.

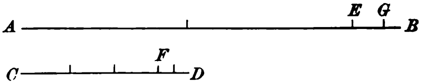
Two quantities are said to be *commensurable* when there is a third quantity of the same kind which is contained an integral number of times in each, and the third quantity is called a *common measure* of these two quantities. The ratio of the two quantities is called a *commensurable* ratio. For example, two lines whose lengths are $3\frac{1}{3}$ inches and $4\frac{3}{5}$ inches are commensurable; a line $\frac{1}{15}$ of an inch in length is contained in the former 50 times and in the latter 69 times.

Two quantities are said to be *incommensurable* when no quantity of the same kind can be found which is contained an integral number of times in each, and the ratio of the two quantities is called an *incommensurable* ratio. For example, suppose a and b to be two lines such that $\frac{a}{b} = \sqrt{2}$. Since the value of $\sqrt{2}$ cannot be expressed exactly either as a decimal or a common fraction, it is impossible to find a line, however small, which will be contained an integral number of times in both a and b. It is possible, however, by taking a very small unit of measure to find a fraction that shall differ from the true value of the ratio by a very small amount, and the smaller the unit of measure, the greater the degree of accuracy.

The ratio of two quantities of the same kind, whether commensurable or incommensurable, is the same as the ratio of their numerical measures referred to the same unit. If x is the unit, and if x is contained m times in a and n times in b, then $\frac{a}{b} = \frac{mx}{nx} = \frac{m}{n}$.

Proposition 74. Problem.

90. *To find the numerical ratio of two straight lines.*



To find the numerical ratio of AB to CD.

Apply CD to AB as many times as possible. Suppose it is contained twice, with a remainder EB.

Then apply EB to CD as many times as possible. Suppose it is contained three times, with a remainder FD.

Then apply FD to EB as many times as possible. Suppose it is contained once, with a remainder GB.

Then apply GB to FD as many times as possible. Suppose it is contained exactly twice in FD.

Then GB is a common measure of AB and CD.

$$FD = 2\,GB.$$

$$EB = EG + GB = FD + GB = 3\,GB.$$

$$CD = CF + FD = 3\,EB + FD = 11\,GB.$$

$$AB = AE + EB = 2\,CD + EB = 25\,GB.$$

$$\therefore \frac{AB}{CD} = \frac{25\,GB}{11\,GB} = \frac{25}{11}.$$

Q.E.F.

Scholium. This process is the same as that used in Arithmetic for finding the greatest common measure of two numbers. GB is not only a common measure of the two lines, but it is also the greatest common measure.

When the two given lines are incommensurable, of course no common measure can be found, and their ratio can be determined only approximately, but the difference between the approximate value and the true value may be made as small as we choose.

The numerical ratio and the greatest common measure of two arcs in the same circle, or in equal circles, may be determined by a similar process.

PROPORTION.

91. An expression of equality between two ratios is called a *proportion*, and the four terms are called *proportionals.* If the ratio of a to b is equal to the ratio of c to d, the proportion may be expressed in either of the following forms:

$$\frac{a}{b}=\frac{c}{d},\quad a:b=c:d,\quad \text{or}\quad a:b::c:d.$$

They may be read, "the ratio of a to b equals the ratio of c to d," or "a is to b as c is to d."

The first and fourth terms of a proportion are called the *extremes*, and the second and third terms are called the *means.* The fourth term is said to be a *fourth proportional* to the other three.

When the means of a proportion are equal, either mean is said to be a *mean proportional* between the first and last terms, and the last term is said to be a *third proportional* to the first and second terms.

A series of equal ratios is called a *continued proportion.*

Four quantities are said to be *reciprocally proportional* when the ratio of the first two is equal to the ratio of the reciprocals of the last two. For example, a, b, c, and d are reciprocally proportional when $a:b::\frac{1}{c}:\frac{1}{d}$; reducing the second ratio to its simplest form, the proportion may be written $a:b::d:c$. Hence, four quantities are reciprocally proportional when the ratio of the first to the second is equal to the ratio of the fourth to the third.

When the means of a proportion are interchanged, the four quantities are said to be in proportion by *alternation.*

When the terms of each ratio of a proportion are interchanged, the four quantities are said to be in proportion by *inversion.*

When the sum of the terms of the first ratio of a proportion is to either term of that ratio as the sum of the terms of the second ratio is to the corresponding term of that ratio, the four quantities are said to be in proportion by *composition.*

When the difference of the terms of the first ratio of a proportion is to either term of that ratio as the difference of the terms of the second ratio is to the corresponding term of that ratio, the four quantities are said to be in proportion by *division.*

When the sum of the terms of the first ratio of a proportion is to their difference as the sum of the terms of the second ratio is to their difference, the four quantities are said to be in proportion by *composition and division.*

PRINCIPLES OF PROPORTION.

92. *In any proportion, the product of the extremes is equal to the product of the means.*

Let the proportion be $a : b :: c : d.$

Then $$\frac{a}{b} = \frac{c}{d}.$$

Multiplying both members of the equation by bd,

$$ad = bc.$$

NOTE. In any product of two quantities in Geometry, we mean simply the product of the numerical measures of the quantities.

93. *A mean proportional between two quantities is equal to the square root of their product.*

Let the proportion be $a : b :: b : c.$

Then by § 92, $$b^2 = ac.$$

Extracting the square root of both members,

$$b = \sqrt{ac}.$$

94. *If the product of two quantities is equal to the product of two other quantities, the factors of either product may be made the means, and the other two the extremes, of a proportion.*

Let $ad = bc.$

Dividing both members of the equation by bd,

$$\frac{a}{b} = \frac{c}{d}.$$

That is, $a : b :: c : d.$

95. *If four quantities are in proportion, they are in proportion by alternation.*

Let the proportion be $a : b :: c : d.$

Then $\frac{a}{b} = \frac{c}{d}.$

Multiplying both members of the equation by $\frac{b}{c}$,

$$\frac{a}{c} = \frac{b}{d}.$$

That is, $a : c :: b : d.$

96. *If four quantities are in proportion, they are in proportion by inversion.*

Let the proportion be $a : b :: c : d.$

Then $\frac{a}{b} = \frac{c}{d}.$

Dividing unity by both members of the equation,

$$\frac{b}{a} = \frac{d}{c}.$$

That is, $b : a :: d : c.$

97. *If four quantities are in proportion, they are in proportion by composition.*

Let the proportion be $a : b :: c : d$.

Then $$\frac{a}{b} = \frac{c}{d}.$$

Adding unity to both members of the equation,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

That is, $a + b : b :: c + d : d$.

The student should prove that $a + b : a :: c + d : c$.

98. *If four quantities are in proportion, they are in proportion by division.*

Let the proportion be $a : b :: c : d$.

Then $$\frac{a}{b} = \frac{c}{d}.$$

Subtracting unity from both members of the equation, and simplifying as in § 97,

$$a - b : b :: c - d : d.$$

The student should prove that $a - b : a :: c - d : c$.

99. *If four quantities are in proportion, they are in proportion by composition and division.*

Let the proportion be $a : b :: c : d$.

Then by § 97, $$\frac{a+b}{b} = \frac{c+d}{d}.$$

By § 98, $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Dividing the first equation by the second,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

That is, $a + b : a - b :: c + d : c - d$.

100. *In a continued proportion, the sum of any number of antecedents is to the sum of the corresponding consequents as any antecedent is to its consequent.*

Let the continued proportion be $a : b = c : d = e : f$.

Then by § 92, $ad = bc$,

and $af = be$.

Also, $ab = ab$.

By adding these equalities,

$$ab + ad + af = ab + bc + be,$$

or $$a(b + d + f) = b(a + c + e).$$

Then by § 94, $a + c + e : b + d + f :: a : b$.

101. *Equimultiples of two quantities are in the same ratio as the quantities themselves.*

Let a and b be the two quantities.

Since both numerator and denominator may be multiplied by the same number without altering the value of the fraction,

$$\frac{ma}{mb} = \frac{a}{b}.$$

That is, $ma : mb :: a : b$.

102. *If four quantities are in proportion, like powers or like roots of these quantities are in proportion.*

Let the proportion be $a : b :: c : d$.

Then $$\frac{a}{b} = \frac{c}{d}.$$

Raising both members of the equation to the nth power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n}.$$

That is, $a^n : b^n :: c^n : d^n$.

Similarly, it may be shown that $\sqrt[n]{a} : \sqrt[n]{b} :: \sqrt[n]{c} : \sqrt[n]{d}$.

103. *If the corresponding terms of two or more proportions are multiplied together, the products are in proportion.*

Let the proportions be $a : b : : c : d$,

and $e : f : : g : h$.

Then $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$.

Multiplying these equalities together,

$$\frac{ae}{bf} = \frac{cg}{dh}.$$

That is, $ae : bf : : cg : dh$.

104. *If the first three terms of a proportion are equal respectively to the first three terms of another proportion, the fourth terms are also equal.*

Let the proportions be $a : b : : c : d$,

and $a : b : : c : e$.

Then by § 92, $ad = bc$, and $ae = bc$.

$$\therefore ad = ae.$$

Dividing both members of the equation by a,

$$d = e.$$

THEORY OF LIMITS.

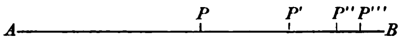
105. A quantity which remains unchanged in value throughout the same discussion is called a *constant quantity* or *constant.* A quantity which may assume different values in the same discussion, according to the conditions imposed upon it, is called a *variable.*

When the value of a variable, which changes according to some fixed law, can be made to have a value as near as we please to a certain constant, but can never become actually equal to it, the constant is called the *limit* of the variable, and the variable is said to *approach* its limit.

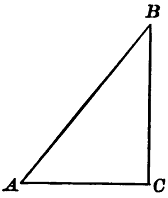
The difference between a variable and its limit is another variable which has zero for its limit.

If a variable is increasing in value, the limit is called a *superior* limit; if the variable is decreasing in value, the limit is called an *inferior* limit.

For example, suppose a point P to move along the line AB in such a way that during the first second it shall move one half the distance from A to B, and that during each successive second it shall move one half the remaining distance. The point P at the end of the successive seconds will occupy the positions P, P', P'', P''', and so on indefinitely. It is evident that the moving point can be made to approach B as near as we please, but it can never actually reach it. Then the distance from A to the moving point is an increasing variable which approaches AB as its limit, and the distance from the moving point to B is a decreasing variable which approaches zero as its limit.

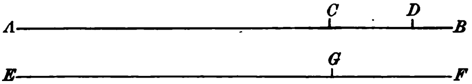


For further illustration, let us consider the right triangle ABC. Suppose the vertex B to move toward the vertex C; then the angle A will approach zero as its limit. If the angle A should actually become equal to zero, AB and AC would coincide, and the triangle would vanish. Furthermore, suppose the vertex B to move away from the vertex C; then the angle A will approach a right angle as its limit. If the angle A should actually become a right angle, AB and CB would be parallel, and the triangle would vanish.



Proposition 75. Theorem.

106. *If two variables are always equal, and each approaches a limit, their limits are equal.*



Hypothesis. AC and EG are two variables which are always equal, and which approach AB and EF respectively as limits.

Conclusion. $AB = EF$.

Proof. If $AB > EF$, some part of AB must be equal to EF. Suppose $AD = EF$.

Then since the variable AC may approach as near as we please to AB, it may have a value larger than AD, or its equal EF. Hence it must be larger than EG which is always less than EF. But this is contrary to the hypothesis that the two variables are always equal.

$$\therefore AB \text{ cannot be } > EF.$$

In like manner it may be proved that EF cannot be $> AB$.

$$\therefore AB = EF.$$

Q.E.D.

SCHOLIUM. The above proof applies to increasing variables which have superior limits. The student should give a similar proof for decreasing variables which have inferior limits.

Proposition 76. Theorem.

107. *If two variables are in a constant ratio, and each approaches a limit, their limits are in the same ratio.*

Hypothesis. a and b are two variables, which approach c and d respectively as limits, and which have a constant ratio such that $\frac{a}{b} = r$.

Conclusion. $\frac{c}{d} = r$.

Proof.

$a = rb$. (?)

The limit of a is c, (?)

and the limit of rb is rd. (?)

$\therefore c = rd$. (?)

$\therefore \frac{c}{d} = r$. (?)

Q.E.D.

LINEAR PROPORTIONS.

108. If a point is taken in a given line, or in the line produced, the distances from the point to the extremities of the line are called the *segments* of the line. If the point is in the line, the line is said to be divided *internally*, and the line is the sum of the two segments. If the point is in the prolongation of the line, the line is said to be divided *externally*, and the line is the difference of the two segments. A line may be divided internally at several points, and the segments of the line are the parts into which it is divided. In general, a line is supposed to be divided internally unless it is stated otherwise.

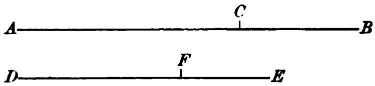
If a line is divided into two segments such that one of the segments is a mean proportional between the whole line and the other segment, the line is said to be divided *in extreme and mean ratio.*

If a line is divided both internally and externally into segments having the same ratio, the line is said to be divided *harmonically.*

Two lines are said to be divided *proportionally* when the segments of one line are in the same ratio as the corresponding segments of the other.

Proposition 77. Theorem.

109. *If two lines are divided proportionally, the ratio of either line to one of its segments is equal to the ratio of the other line to its corresponding segment.*



Hypothesis. The two lines AB and DE are divided at C and F respectively so that $AC : CB :: DF : FE$.

Conclusion. $AB : AC :: DE : DF$, and $AB : CB :: DE : FE$.

Proof. $AC + CB : AC :: DF + FE : DF$,

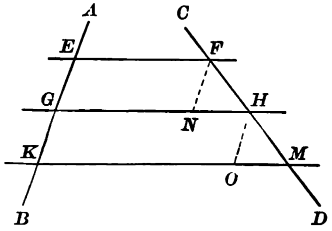
and $AC + CB : CB :: DF + FE : FE$. (?)

$\therefore AB : AC :: DE : DF$, and $AB : CB :: DE : FE$. (?)

Q.E.D.

Proposition 78. Theorem.

110. *If three or more parallel lines intercept equal segments on one of two transversals, they intercept equal segments on the other transversal also.*



Hypothesis. The ∥ lines EF, GH, and KM are cut by the transversals AB and CD, so that $EG = GK$.

Conclusion. $FH = HM$.

Proof. Draw FN and HO ∥ to AB.

Then $FN = EG$, and $HO = GK$. (?)

$\therefore$ FN and NO are equal and ∥. (?)

$\triangle FNH = \triangle HOM$. (?)

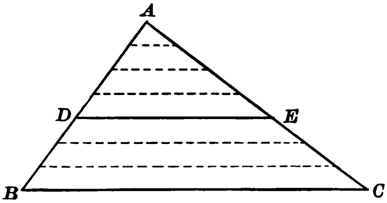
$\therefore$ $FH = HM$. (?)

Q.E.D.

Proposition 79. Theorem.

111. *If a line is drawn through two sides of a triangle parallel to the third side, it divides the two sides proportionally.*

CASE I. *When the segments of either side are commensurable.*



Hypothesis. In the $\triangle ABC$, DE is drawn ∥ to BC, and AD and DB are commensurable.

Conclusion. $AD : DB :: AE : EC$.

Proof. Suppose a common measure of AD and DB to be contained in AD m times and in DB n times.

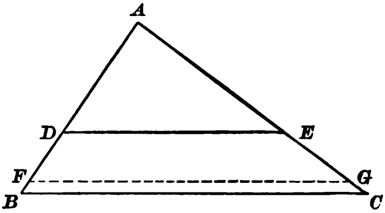
$$\text{Then } \frac{AD}{DB} = \frac{m}{n}.$$

Through the several points of division on AD and DB draw lines ∥ to BC. These lines divide AE into m parts and EC into n parts, all of which are equal. (?)

$$\therefore \frac{AE}{EC} = \frac{m}{n}.$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \quad (?)$$

CASE II. *When the segments of either side are incommensurable.*



Hypothesis. In the $\triangle ABC$, DE is drawn $\parallel$ to BC, and AD and DB are incommensurable.

Conclusion. $AD : DB :: AE : EC$.

Proof. Divide AD into any number of equal parts, and apply one of these parts to DB as a unit of measure. It will be contained a certain number of times with a remainder FB, which is less than the unit of measure.

Draw $FG \parallel$ to BC.

$$\text{Then } \frac{AD}{DF} = \frac{AE}{EG}. \qquad \text{(CASE I.)}$$

If, now, the number of parts into which AD is divided is indefinitely increased, the unit of measure is indefinitely decreased, and the point F comes indefinitely near to the point B.

Then the variables $\frac{AD}{DF}$ and $\frac{AE}{EG}$ continue equal, and their limits are $\frac{AD}{DB}$ and $\frac{AE}{EC}$ respectively.

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \qquad (?)$$

Q.E.D.

COR. *If a line is drawn through two sides of a triangle parallel to the third side, the ratio of either side to one of its segments is equal to the ratio of the other side to its corresponding segment.*

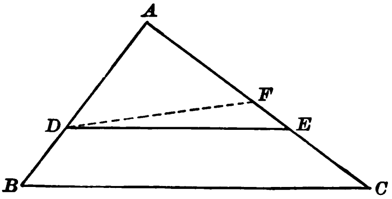
Proposition 80. Theorem.

112. *If two lines are cut by three or more parallel lines, the corresponding segments of the two lines are proportional.*

Use the method of proof given in Prop. 79.

Proposition 81. Theorem.

113. *If a line divides two sides of a triangle proportionally, it is parallel to the third side.*



Hypothesis. In the $\triangle ABC$, DE is drawn so that

$$AD : DB :: AE : EC.$$

Conclusion. DE is $\parallel$ to BC.

Proof. From D draw $DF \parallel$ to BC.

Then $\frac{AB}{AD} = \frac{AC}{AF}$. (?)

But $\frac{AB}{AD} = \frac{AC}{AE}$. (?)

$\therefore AF = AE$. (?)

$\therefore DE$ coincides with DF. (?)

$\therefore DE$ is $\parallel$ to BC. (?)

Q.E.D.

Cor. *If a line divides two sides of a triangle so that the ratio of either side to one of its segments is equal to the ratio of the other side to its corresponding segment, the line is parallel to the third side.*

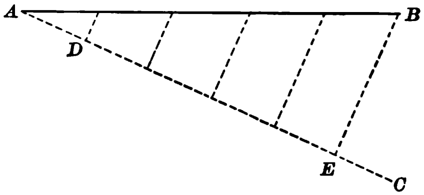
Ex. 229. From E, a point in the common base of the two triangles ABC and ABD, straight lines are drawn parallel to AC and AD, meeting BC and BD at F and G respectively; prove that FG is parallel to CD.

Ex. 230. A line is drawn from a given point to a given straight line which does not pass through the point; find the locus of the point which divides the line in a given ratio.

Ex. 231. From a point in the base of a triangle straight lines are drawn parallel to the other two sides; find the locus of the point of intersection of the diagonals of the parallelogram thus formed.

Proposition 82. Problem.

114. *To divide a given straight line into any number of equal parts.*



To divide the line AB into m equal parts.

Construction. From A draw any indefinite line AC.

On AC lay off m equal distances, each equal to any convenient length AD.

Through E, the last point of division on AC, draw EB, and through the other points of division of AC draw lines ∥ to EB.

These lines divide AB into m equal parts.

[The proof is left to the student.]

Ex. 232. To construct an equilateral triangle when the perimeter is given.

Proposition 83. Problem.

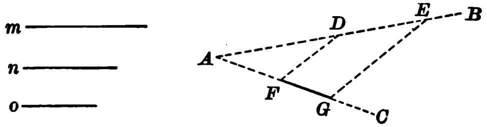
115. *To divide a given straight line into parts proportional to any number of given lines.*

The construction is similar to that of Prop. 82. Instead of taking equal divisions on the line AC, take divisions equal to the given lines.

Ex. 233. To construct a triangle when the perimeter is given, and the sides are proportional to three given lines.

Proposition 84. Problem.

116. *To find a fourth proportional to three given straight lines.*



To find a fourth proportional to the lines m, n, and o.

Construction. Draw the indefinite lines AB and AC, making any convenient ∠ with each other.

On AB lay off $AD = m$ and $DE = n$. On AC lay off $AF = o$.

Draw DF, and draw $EG \parallel$ to DF.

Then FG is the required fourth proportional.

[The proof is left to the student.]

Ex. 234. To produce a line AB to a point C so that $AB : AC :: m : n$, m being smaller than n.

Ex. 235. To construct two straight lines when their sum and their ratio are given.

Ex. 236. To construct two straight lines when their difference and their ratio are given.

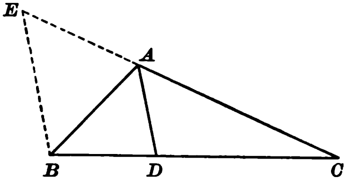
Proposition 85. Problem.

117. *To find a third proportional to two given straight lines.*

The construction is similar to that of Prop. 84. *DE* and *AF* should both be taken equal to the second line.

Proposition 86. Theorem.

118. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.*



Hypothesis. In the $\triangle ABC$, AD is the bisector of the $\angle CAB$.

Conclusion. $BD : DC :: AB : AC$.

Proof. Draw $BE \parallel$ to DA to meet CA produced at E.

$$\angle E = \angle CAD. \qquad (?)$$

$$\angle ABE = \angle DAB. \qquad (?)$$

$$\therefore\ \angle E = \angle ABE. \qquad (?)$$

$$\therefore\ AB = AE. \qquad (?)$$

$$BD : DC :: AE : AC. \qquad (?)$$

$$\therefore\ BD : DC :: AB : AC. \qquad (?)$$

Q.E.D.

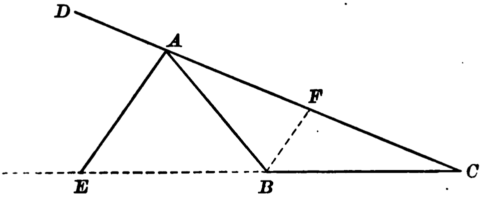
Ex. 237. State and prove the converse of Prop. 86.

Ex. 238. To divide one side of a given triangle into segments proportional to the adjacent sides.

Ex. 239. The side BC of a triangle ABC is bisected at D, and the angles ADB and ADC are bisected by the straight lines DE and DF, meeting AB and AC at E and F respectively; prove that EF is parallel to BC.

Proposition 87. Theorem.

119. *The bisector of an exterior angle of a triangle divides the opposite side externally into segments which are proportional to the adjacent sides.*



Hypothesis. In the $\triangle ABC$, AE is the bisector of the exterior $\angle BAD$.

Conclusion. $EB : EC :: AB : AC$.

Proof. Draw $BF \parallel$ to EA.

[To be completed by the student.]

Ex. 240. State and prove the converse of Prop. 87.

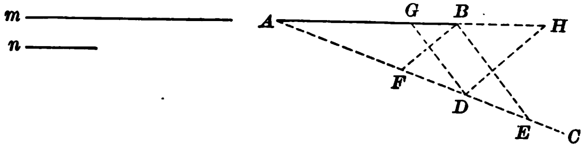
Proposition 88. Theorem.

120. *The bisector of an angle of a triangle and the bisector of the adjacent exterior angle divide the opposite side harmonically.*

Consult Prop. 86, Prop. 87, and the third paragraph of § 108.

Proposition 89. Problem.

121. *To divide a given straight line harmonically in a given ratio.*



To divide the line AB harmonically in the ratio $m : n$.

Construction. From A draw any indefinite line AC.

On AC lay off $AD = m$, and from D lay off $DE = n$ and $DF = n$.

Draw EB and FB; then draw $DG \parallel$ to EB and $DH \parallel$ to FB.

The line AB is divided harmonically at G and H in the ratio $m : n$.

[The proof is left to the student.]

Ex. 241. Solve Prop. 89 by applying Prop. 88.

Ex. 242. If the line AB is divided harmonically at C and D, the line CD is divided harmonically at A and B.

Ex. 243. If the line AB is divided harmonically at C and D, and E is the middle point of AB, AE is a mean proportional between CE and DE.

PROBLEMS OF COMPUTATION.

1. Find the fourth proportional to the lines whose lengths are 32, 40, and 44.

2. Find the third proportional to the lines whose lengths are 18 and 32.

3. Find the mean proportional between the lines whose lengths are 18 and 32.

4. In the triangle ABC, DE is drawn parallel to BC; the lengths of AB, AC, and AD are 24, 18, and 16 respectively. Find the length of EC.

5. In the triangle ABC, $AB = 192$ ft. If a line drawn parallel to BC divides AC in the ratio $5:3$, what are the lengths of the segments of AB?

6. The sides of a triangle are 12, 16, and 20. Find the lengths of the segments of each side formed by the bisectors of the opposite angles.

7. The sides of a triangle are a, b, and c. Find the lengths of the segments of c formed by the bisector of the opposite angle.

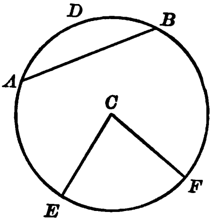
8. If the acute angles of a right triangle are 30° and 60°, what is the ratio of the segments into which the bisector of the angle of 60° divides the opposite side?

BOOK III.

THE CIRCLE.

122. [See definitions in § 6.]

A straight line joining any two points of the circumference of a circle is called a *chord;* as *AB.* Every chord, therefore, divides the circumference into two arcs, which are called *conjugate* arcs; as *ADB* and *AEB.* When two conjugate arcs are unequal, the greater is called the *major* arc, and the smaller is called the *minor* arc. The chord is said to *subtend* the arc, and the arc is said to be *subtended* by the chord. When a chord and its subtended arc are mentioned, the minor arc is always understood unless it is otherwise stated.

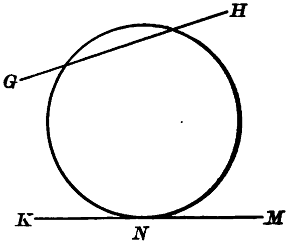


The portion of a circle bounded by a chord and its subtended arc is called a *segment;* as *ADB.* A segment equal to one half the circle is called a *semicircle.*

The portion of a circle bounded by an arc and the radii drawn to its extremities is called a *sector;* as *ECF.*

A straight line which intersects the circumference in two points is called a *secant;* as *GH.*

A straight line which has one point in common with the circumference, but does not meet it again, however far it is produced, is said to be *tangent* to the circle, or to *touch* it. The line is called a *tangent,* and the common point is called the *point of contact,* or



point of tangency. For example, *KM* is a tangent, and *N* is the point of contact.

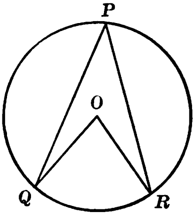
A common tangent to two circles which intersects the line joining the centres is called an *interior* common tangent. A common tangent which does not intersect the line joining the centres is called an *exterior* common tangent.

NOTE. The length of a secant drawn from a point without a circle is the distance from the point to the farther point of intersection.

The length of a tangent is the distance from the external point to the point of contact.

Two circles are said to be *tangent* to each other, or to *touch* each other, when their circumferences have one point in common, and only one. When each of the circles is outside the other, they are said to be tangent *externally;* when one circle is within the other, they are said to be tangent *internally.*

An angle whose sides are chords, and whose vertex lies in the circumference, is called an *inscribed angle;* as *QPR.* An angle is said to be *inscribed in a segment* when its vertex lies in the arc of the segment and its sides are chords drawn to the extremities of the arc. An angle whose sides are radii, and whose vertex is at the centre, is called a *central angle;* as *QOR.* A central angle is said to *intercept* an arc, and the arc is said to *subtend* the central angle.



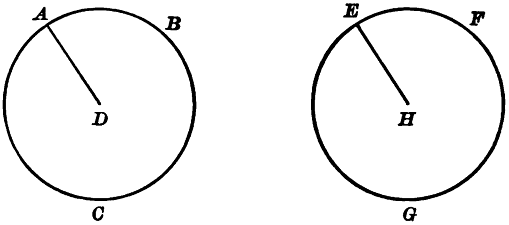
A polygon is said to be *inscribed in a circle* when all its sides are chords of the circle. The circle is then said to be *circumscribed about* the polygon.

A polygon is said to be *circumscribed about a circle* when all its sides are tangent to the circle. The circle is then said to be *inscribed in* the polygon.

Circles which have the same centre are said to be *concentric.*

Proposition 90. Theorem.

123. *Two circles are equal when they have equal radii.*



Hypothesis. In the ⓢ ABC and EFG, radius $DA =$ radius HE.

Conclusion. $\odot ABC = \odot EFG$.

Proof. Apply the $\odot ABC$ to the $\odot EFG$ so that the centre D shall fall on the centre H.

Then all the points on the circumference ABC will fall on the circumference EFG. (?)

$$\therefore \odot ABC = \odot EFG. \qquad (?)$$

Q.E.D.

Cor. I. *If two circles are equal, their radii are equal.*

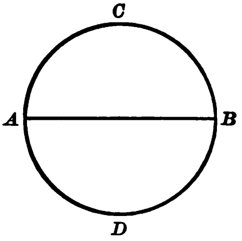
Cor. II. *If two concentric circles have equal radii, their circumferences coincide.*

Scholium. Since two equal circles will coincide when one is applied to the other, a theorem containing the expression "in the same circle, or in equal circles," may be proved by using either a figure containing one circle or a figure containing two circles.

Ex. 244. Find the locus of the centre of a circle which has a given radius and whose circumference passes through a given point.

Proposition 91. Theorem.

124. *Any diameter bisects the circle and its circumference.*



Hypothesis. AB is a diameter of the $\odot ACBD$.

Conclusion. Segment $ACB =$ segment ADB, and arc $ACB =$ arc ADB.

Proof. Let the segment ACB revolve on AB as an axis until it falls on the segment ADB.

Then all points on the arc ACB will fall on the arc ADB. (?)

$\therefore$ Arc $ACB =$ arc ADB,

and segment $ACB =$ segment ADB. (?)

Q.E.D.

Proposition 92. Theorem.

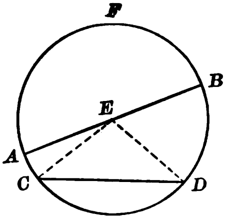
125. *A straight line cannot intersect the circumference of a circle in more than two points.*

Hint. Suppose the line could intersect the circle in three points. Draw radii to these points, and consult Prop. 14, Cor.

Ex. 245. If two circumferences intersect each other, the distance between their centres is less than the sum and greater than the difference of their radii.

Proposition 93. Theorem.

126. *The diameter of a circle is longer than any other chord.*



Hypothesis. In the ⊙ AFD, of which E is the centre, AB is a diameter, and CD is any other chord.

Conclusion. $AB > CD$.

Proof. Draw the radii EC and ED.

$EC + ED > CD$. (?)

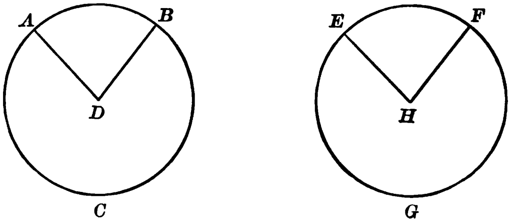
But $EC + ED = AB$. (?)

$\therefore AB > CD$. (?)

Q.E.D.

Proposition 94. Theorem.

127. *In the same circle, or in equal circles, equal central angles intercept equal arcs.*



Hypothesis. In the equal Ⓢ ABC and EFG, of which D and H are the respective centres, $\angle D = \angle H$.

Conclusion. Arc AB = arc EF.

Proof. $DA = HE$. (?)

Apply the ⊙ ABC to the ⊙ EFG so that DA shall coincide with HE, D falling on H, and A on E.

Then DB will fall along HF. (?)

The point B will fall on the point F. (?)

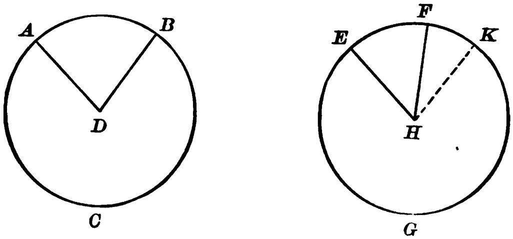
∴ the arc AB will coincide with the arc EF. (?)

∴ arc AB = arc EF. (?)

Q.E.D.

Proposition 95. Theorem.

128. *In the same circle, or in equal circles, if two central angles are unequal, the greater angle intercepts the greater arc.*



Hypothesis. In the equal Ⓢ ABC and EFG, of which D and H are the respective centres, $\angle D > \angle EHF$.

Conclusion. Arc AB > arc EF.

Proof. Draw the radius HK making $\angle EHK = \angle D$.

Then arc EK = arc AB. (?)

$\angle EHK > \angle EHF$. (?)

∴ arc EK > arc EF. (?)

∴ arc AB > arc EF. (?)

Q.E.D.

Proposition 96. Theorem.

129. *In the same circle, or in equal circles, equal arcs subtend equal central angles.*

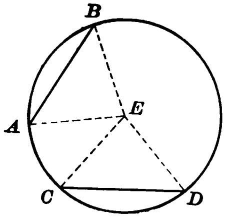
Use either the method of superposition or the indirect method.

Proposition 97. Theorem.

130. *In the same circle, or in equal circles, if two arcs are unequal, the greater arc subtends the greater central angle.*

Proposition 98. Theorem.

131. *In the same circle, or in equal circles, equal chords subtend equal arcs.*



Hypothesis. In the ⊙ ABD, of which E is the centre, chord $AB =$ chord CD.

Conclusion. Arc $AB =$ arc CD.

Proof. Draw the radii EA, EB, EC, and ED.

$\triangle ABE = \triangle CDE$. (?)

$\therefore \angle AEB = \angle CED$. (?)

$\therefore$ arc $AB =$ arc CD. (?)

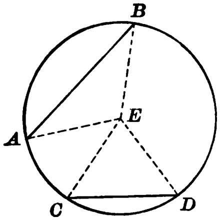
Q.E.D.

Proposition 99. Theorem.

132. *In the same circle, or in equal circles, equal arcs are subtended by equal chords.*

Proposition 100. Theorem.

133. *In the same circle, or in equal circles, if two chords are unequal, the greater chord subtends the greater arc.*



Hypothesis. In the $\odot ABD$, of which E is the centre, chord $AB >$ chord CD.

Conclusion. Arc $AB >$ arc CD.

Proof. Draw the radii EA, EB, EC, and ED.

$EA = EC$, and $EB = ED$. (?)

$\angle AEB > \angle CED$. (?)

$\therefore$ arc $AB >$ arc CD. (?)

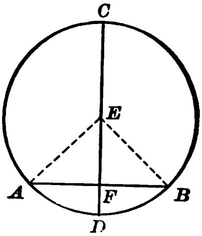
Q.E.D.

Proposition 101. Theorem.

134. *In the same circle, or in equal circles, if two arcs are unequal, the greater arc is subtended by the greater chord.*

Proposition 102. Theorem.

135. *The diameter perpendicular to a chord bisects the chord and the arcs subtended by it.*



Hypothesis. In the $\odot ACB$, of which E is the centre, the diameter CD is $\perp$ to the chord AB at F.

Conclusion. $AF = FB$, arc $AD =$ arc DB, and arc $AC =$ arc CB.

Proof. Draw the radii EA and EB.

$\triangle EFA = \triangle EFB$. (?)

$\therefore AF = FB$. (?)

$\angle AED = \angle DEB$. (?)

$\therefore$ arc $AD =$ arc DB. (?)

Arc $CAD =$ arc CBD. (?)

$\therefore$ arc $AC =$ arc CB. (?)

Q.E.D.

Cor. *The perpendicular drawn from the centre of a circle to a chord bisects the chord.*

Proposition 103. Theorem.

136. *The perpendicular bisector of a chord passes through the centre of the circle, and bisects the subtended arcs.*

Consult Prop. 16, Cor. I.

Proposition 104. Theorem.

137. *The diameter which bisects a chord is perpendicular to the chord, and bisects the subtended arcs.*

Proposition 105. Theorem.

138. *A radius drawn to the middle point of an arc is the perpendicular bisector of the chord which subtends the arc.*

Ex. 246. The straight line drawn from the middle point of a chord to the middle point of its subtended arc is perpendicular to the chord.

Ex. 247. The line joining the middle points of two parallel chords of a circle passes through the centre.

Ex. 248. The segments of any straight line intercepted between the circumferences of two concentric circles are equal.

Ex. 249. A straight line drawn through a point of intersection of two circles has an extremity on each circumference. Prove that the segment of the line included between perpendiculars let fall from the centres of the circles is equal to one half the line.

Ex. 250. Find the locus of the centre of a circle whose circumference passes through two given points.

Ex. 251. Find the locus of the middle points of a system of parallel chords drawn in a circle.

Ex. 252. Through a given point within a given circle to draw a chord which shall be bisected at the point.

Ex. 253. To construct a circle which shall have a given radius, and whose circumference shall pass through two given points.

Ex. 254. To construct a circle which shall have its centre in a given line, and whose circumference shall pass through two given points without the line.

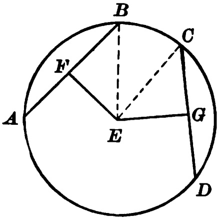
Proposition 106. Problem.

139. *To bisect a given arc.*

The construction is the same as that of Prop. 17.

Proposition 107. Theorem.

140. *In the same circle, or in equal circles, equal chords are equally distant from the centre.*



Hypothesis. In the ⊙ ABD, chord $AB =$ chord CD, and EF and EG are the ⊥ distances from the centre to AB and CD respectively.

Conclusion. $EF = EG$.

Proof. Draw the radii EB and EC.

$FB = GC$. (?)

Then $\triangle EFB = \triangle EGC$. (?)

$\therefore\ EF = EG$. (?)

Q.E.D.

Proposition 108. Theorem.

141. *In the same circle, or in equal circles, chords equally distant from the centre are equal.*

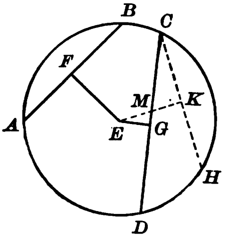
Ex. 255. The chords cut off by two equal circles from a straight line parallel to the line joining their centres or from a straight line passing through the middle of the line joining their centres are equal.

Ex. 256. To draw a chord in a circle which shall be equal to a given chord and parallel to a given straight line.

Ex. 257. Find the locus of the middle point of a chord of given length drawn in a given circle.

Proposition 109. Theorem.

142. *In the same circle, or in equal circles, if two chords are unequal, the less chord is at the greater distance from the centre.*



Hypothesis. In the ⊙ ABC, chord $AB <$ chord CD, and EF and EG are the $\perp$ distances from the centre to AB and CD respectively.

Conclusion. $EF > EG$.

Proof. Draw the chord $CH = AB$, and draw $EK \perp$ to CH.

Then chord $CH <$ chord CD. (?)

Arc $CH <$ arc CD. (?)

Hence the point H falls on the arc CD.

Then every point of the chord CH is on the opposite side of CD from the centre, and EK must intersect CD.

Let M be the point of intersection.

$EM > EG$. (?)

$EK > EM$. (?)

Much more is $EK > EG$.

But $EF = EK$. (?)

$\therefore EF > EG$. (?)

Q.E.D.

Proposition 110. Theorem.

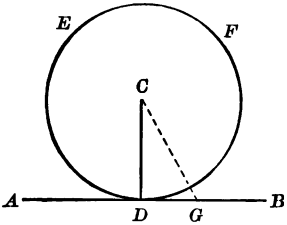
143. *In the same circle, or in equal circles, if two chords are unequally distant from the centre, the chord at the less distance is the greater.*

Use the indirect method.

Ex. 258. The shortest chord which can be drawn through a given point within a circle is the chord perpendicular to the diameter passing through the point.

Proposition 111. Theorem.

144. *A straight line perpendicular to a radius at its extremity is a tangent to the circle.*



Hypothesis. CD is a radius of the ⊙ EFD, and AB is ⊥ to CD at its extremity D.

Conclusion. AB is a tangent to the ⊙.

Proof. Draw CG, any other line from C to the line AB.

Then $CG > CD$. (?)

∴ the point G is without the ⊙.

Hence every point of the line AB, except D, is without the ⊙.

∴ AB is a tangent to the ⊙. (?)

Q.E.D.

Ex. 259. A straight line oblique to a radius at its extremity intersects the circumference of the circle.

Proposition 112. Theorem.

145. *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*

HINT. Prove that the radius is the shortest line that can be drawn from the centre to the tangent.

Proposition 113. Theorem.

146. *A perpendicular to a tangent at the point of contact passes through the centre of the circle.*

HINT. Draw a radius to the point of contact, and consult Prop. 112 and Prop. 1.

COR. *A perpendicular drawn from the centre of a circle to a tangent passes through the point of contact.*

Ex. 260. The tangents to a circle at the extremities of a diameter are parallel.

Ex. 261. In any circle, a chord parallel to a tangent is bisected by the diameter drawn to the point of contact.

Ex. 262. In two concentric circles, all chords of the greater which are tangent to the smaller are equal and are bisected at the point of contact.

Ex. 263. In two concentric circles, if a chord of the greater is tangent to the smaller, every chord of the greater having the same length is tangent to the smaller.

Ex. 264. Find the locus of the centre of a circle which touches a given line at a given point.

Ex. 265. Find the locus of the centre of a circle which has a given radius and touches a given line.

Ex. 266. Find the locus of the centre of a circle which touches two given parallel lines.

Ex. 267. Find the locus of the centre of a circle which touches two given intersecting lines.

Ex. 268. Find the locus of the extremity of a tangent of given length drawn to a given circle.

Proposition 114. Problem.

147. *Through a given point on the circumference of a given circle, to draw a tangent to the circle.*

Consult Prop. 111.

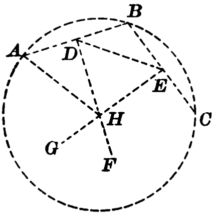
Ex. 269. To draw a tangent to a given circle which shall be parallel to a given straight line.

Ex. 270. To draw a tangent to a given circle which shall make a given angle with a given straight line.

Ex. 271. If a tangent to a circle is parallel to a chord, the point of contact is the middle point of the arc subtended by the chord.

Proposition 115. Theorem.

148. *Through any three points not in the same straight line one circumference can be drawn, and only one.*



Hypothesis. A, B, and C are three points not in the same straight line.

Conclusion. One circumference can be drawn through A, B, and C, and only one.

Proof. Draw AB and BC.

Draw DF the ⊥ bisector of AB, and EG the ⊥ bisector of BC; also draw DE.

DF and *EG* will intersect. (?)

Let *H* be the point of intersection.

Then *H* is equally distant from *A*, *B*, and *C*. (?)

Draw *HA*.

Then the circumference described with *H* as a centre and with *HA* as a radius will pass through *A*, *B*, and *C*.

If a second circumference could be drawn through *A*, *B*, and *C*, its centre would lie in both *DF* and *EG*. (?)

But two lines can intersect in only one point. (?)

∴ *H* is the only centre of a circumference passing through *A*, *B*, and *C*.

∴ only one circumference can be drawn through *A*, *B*, and *C*. (?)

Q.E.D.

Cor. I. *A circle can be circumscribed about any triangle.*

Cor. II. *Two circumferences can intersect in only two points.*

Ex. 272. A circle can be circumscribed about a rectangle.

Ex. 273. A circle can be circumscribed about an isosceles trapezoid.

Proposition 116. Problem.

149. *To circumscribe a circle about a given triangle.*

Consult Prop. 115.

Scholium. The same method of solution may be used for the following problems:

To describe a circumference which shall pass through three given points not in the same straight line.

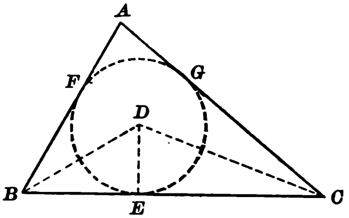
To find the centre of a given circle.

To find the centre of a given arc.

Ex. 274. To construct a circle when a chord and a point on the circumference are given.

Proposition 117. Theorem.

150. *A circle can be inscribed in any triangle.*



Hypothesis. *ABC* is a △.

Conclusion. A ⊙ can be inscribed in the △ *ABC*.

Proof. Draw *BD* and *CD*, bisecting the ∡ *ABC* and *ACB* respectively.

BD and *CD* will intersect. (?)

Let *D* be the point of intersection.

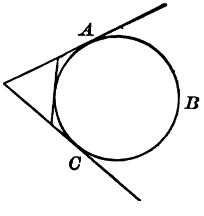
Then *D* is equidistant from *AB*, *AC*, and *BC*. (?)

Draw *DE* ⊥ to *BC*.

Then a ⊙ described with *D* as a centre and with *DE* as a radius will be inscribed in the △ *ABC*. (?)

Q.E.D.

SCHOLIUM. If the sides of a triangle are produced, and the exterior angles thus formed are bisected, the points of intersection of these bisectors will be the centres of three circles, each one of which will touch one side of the triangle and the other two sides produced. These three circles are called *escribed* circles. The circle *ABC* is an example of an escribed circle.



Ex. 275. A circle can be inscribed in a square.

Ex. 276. The bisectors of the three angles of a triangle meet in the centre of the inscribed circle.

Proposition 118. Problem.

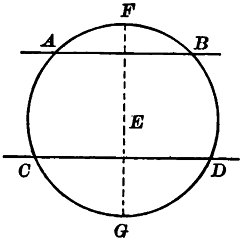
151. *To inscribe a circle in a given triangle.*

Consult Prop. 117.

Proposition 119. Theorem.

152. *Two parallel lines intercept equal arcs on a circumference.*

Case I. *When the parallel lines are both secants.*



Hypothesis. In the ⊙ ABG, AB and CD are ∥ secants intercepting the arcs AC and BD.

Conclusion. Arc $AC =$ arc BD.

Proof. Draw the diameter $FG \perp$ to AB.

Then FG is $\perp$ to CD. (?)

∴ Arc $FA =$ arc FB, and arc $FC =$ arc FD. (?)

Hence arc $AC =$ arc BD. (?)

Case II. *When one of the parallel lines is a secant, and the other a tangent.*

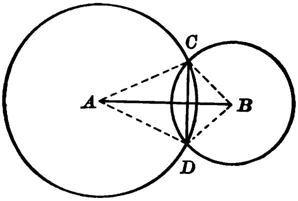
Hint. Draw a diameter from the point of contact, and consult Prop. 112, Prop. 25, and Prop. 102.

Case III. *When the parallel lines are both tangents.*

Hint. Draw a secant ∥ to the tangents, and consult Case II.

Proposition 120. Theorem.

153. *If two circumferences intersect each other, the straight line joining their centres is the perpendicular bisector of their common chord.*



Hypothesis. A and B are the centres of two circumferences which intersect at C and D, and CD is the common chord.

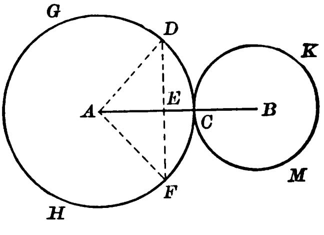
Conclusion. AB is the $\perp$ bisector of CD.

Proof. Draw the radii AC, AD, BC, and BD.

[To be completed by the student.]

Proposition 121. Theorem.

154. *If two circles are tangent to each other, their centres and the point of contact lie in the same straight line.*



Hypothesis. The two ⓢ GCH and KCM, of which A and B are the respective centres, are tangent to each other at C.

Conclusion. A, B, and C lie in the same straight line.

Proof. If possible, suppose that the point of contact lies without the straight line AB at some point D.

Draw $DE \perp$ to AB, and produce it to F, making $EF = DE$; also draw AD and AF.

Then $AF = AD$. (?)

$\therefore$ the point F is on the circumference GCH. (?)

In like manner it may be proved that the point F is on the circumference KCM.

$\therefore$ the two circumferences have two points in common, a conclusion which is contrary to the hypothesis.

$\therefore$ the point of contact does not lie without the line AB, and A, B, and C lie in the same straight line.

Q.E.D.

Cor. *If two circles are tangent to each other, the radii drawn to the point of contact lie in the same straight line.*

Proposition 122. Theorem.

155. *If two circles are tangent to each other, a tangent to one circle at the point of contact is tangent to the other circle also.*

Hint. Draw radii to the point of contact, and consult Prop. 112, Prop. 121, Cor., and Prop. 111.

Scholium. In order to draw a common tangent to two circles at the point of contact, draw a tangent to one of the circles at this point.

Ex. 277. Find the locus of the centre of a circle which is tangent to a given circle at a given point.

Ex. 278. Find the locus of the centre of a circle which has a given radius and is tangent to a given circle.

Ex. 279. With a given point as centre, to construct a circle which shall be tangent to a given circle.

Proposition 123. Theorem.

156. *In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs.*

CASE I. *When the arcs are commensurable.*

CASE II. *When the arcs are incommensurable.*

Use the method of proof given in Prop. 79, and consult Prop. 96.

SCHOLIUM. If a central angle is taken as the unit angle and its intercepted arc as the unit arc, the ratio of any central angle to the unit angle is equal to the ratio of its intercepted arc to the unit arc. For this reason, the numerical measure of a central angle is the same as the numerical measure of its intercepted arc; that is, *a central angle is measured by its intercepted arc.*

The usual unit of measure for arcs is the *degree* (°), which is the arc intercepted by a central angle of one degree, and every circumference consists of 360°. A degree is divided into sixty equal parts called *minutes* ($'$), and a minute is divided into sixty equal parts called *seconds* ($''$). The numerical measure of any arc is expressed in the same number of degrees, minutes, and seconds as its subtended central angle. For example, a central angle of $27° 5' 18''$ intercepts an arc of $27° 5' 18''$.

Ex. 280. The shortest line that can be drawn from a point within a circle to the circumference lies in the radius passing through the point.

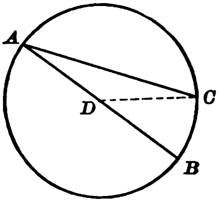
Ex. 281. The longest line that can be drawn from a point within a circle to the circumference passes through the centre of the circle.

Ex. 282. AB and AC are two equal chords of a circle; prove that the straight line which bisects the angle BAC passes through the centre.

Proposition 124. Theorem.

157. *An inscribed angle is measured by one half its intercepted arc.*

Case I. *When one side of the angle is a diameter.*



Hypothesis. In the ⊙ ABC, $\angle A$ is an inscribed $\angle$, and AB is a diameter.

Conclusion. $\angle A$ is measured by $\frac{1}{2}$ arc BC.

Proof. Draw the radius DC.

$$\angle A = \angle C. \qquad (?)$$

$$\angle BDC = \angle A + \angle C. \qquad (?)$$

$$\therefore \angle BDC = 2 \angle A. \qquad (?)$$

$$\therefore \angle A = \tfrac{1}{2} \angle BDC. \qquad (?)$$

But $\angle BDC$ is measured by arc BC. (?)

$\therefore \angle A$ is measured by $\frac{1}{2}$ arc BC.

Case II. *When the centre of the circle is within the angle.*

Hint. Draw a diameter through the vertex of the $\angle$, and consult Case I.

Case III. *When the centre of the circle is without the angle.*

Cor. I. *Angles inscribed in the same segment, or in equal segments, are equal.*

Cor. II. *An angle inscribed in a semicircle is a right angle.*

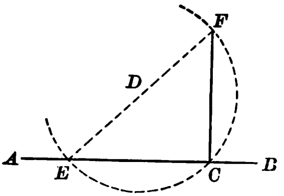
COR. III. *An angle whose vertex lies in the circumference of a circle and whose sides are a chord and the external segment of a secant is measured by one half the sum of the intercepted arcs.*

COR. IV. *The opposite angles of an inscribed quadrilateral are supplementary.*

Ex. 283. Prove by the use of Prop. 124 that the sum of the three angles of a triangle is equal to two right angles.

Proposition 125. Problem.

158. *At a given point in a given straight line, to erect a perpendicular to that line.*



To erect a ⊥ to the line *AB* at *C*, a point in *AB*.

Construction. With any point *D*, not in the line *AB*, as a centre and with the distance *DC* as a radius, describe an arc greater than a semi-circumference, intersecting *AB* at *C* and *E*.

Draw the diameter *EF*; also draw *CF*.

Then *CF* is the required ⊥.

[The proof is left to the student.]

SCHOLIUM. The construction is preferable to that given in Prop. 18 when the given point is at, or near, one extremity of the line.

Ex. 284. If one side of a quadrilateral inscribed in a circle is produced, the exterior angle is equal to the opposite interior angle of the quadrilateral.

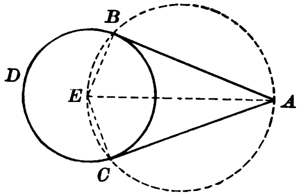
Ex. 285. P is any point in the arc subtended by the chord AB. Prove that the sum of the angles PAB and PBA is constant.

Ex. 286. If the hypotenuse of a right triangle is the diameter of a circle, the circumference passes through the vertex of the right angle.

Ex. 287. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the straight line joining the ends of the diameters passes through the other point of intersection.

Proposition 126. Problem.

159. *From a given point without a given circle, to draw a tangent to the circle.*



To draw a tangent to the ⊙ BCD from A, a point without the ⊙.

Construction. Find E, the centre of the ⊙, and draw EA.

On EA as a diameter, describe a circumference intersecting the given circumference at B and C.

Draw AB and AC.

Either AB or AC is the required tangent.

Proof. Draw EB and EC.

[To be completed by the student.]

SCHOLIUM. This problem always has two solutions.

Proposition 127. Theorem.

160. *The two tangents to a circle drawn from an exterior point are equal.*

Hint. Draw a line from the centre of the circle to the exterior point, and also two radii to the points of contact.

Cor. *If a line is drawn from the centre of a circle to the point of intersection of two tangents, this line bisects both the angle formed by the tangents and the angle formed by the radii drawn to the points of contact.*

Ex. 288. The line drawn from the centre of a circle to the point of intersection of two tangents is the perpendicular bisector of the chord joining the points of contact.

Ex. 289. AB and AC are tangents to the circle whose centre is O; prove that the angle OBC is equal to one half the angle A.

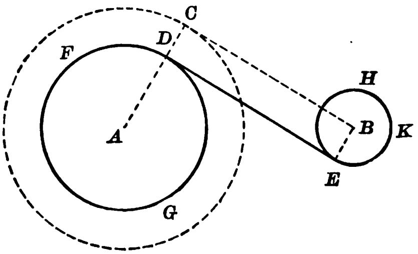
Ex. 290. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides.

Ex. 291. State and prove the converse of Ex. 290.

Proposition 128. Problem.

161. *To draw a common tangent to two given circles.*

Case I. *To draw an interior common tangent.*



To draw an interior common tangent to the Ⓢ FDG and HKE.

Construction. With A, the centre of one ⊙, as a centre and with a radius equal to the sum of the radii of the two Ⓢ, describe a circumference.

From B, the centre of the other ⊙, draw BC tangent to this third ⊙.

Draw AC, meeting the given circumference at D.

Draw $BE \parallel$ to CA, meeting the circumference at E, and draw DE.

Then DE is an interior common tangent.

[The proof is left to the student.]

CASE II. *To draw an exterior common tangent.*

If the Ⓢ are unequal, the construction is similar to that of Case I. Let the radius of the third ⊙ be equal to the difference of the radii of the two given Ⓢ.

If the Ⓢ are equal, draw a line joining the two centres. Draw radii ⊥ to this line, and join their extremities.

SCHOLIUM. If the two circles are wholly exterior to each other, two exterior tangents and two interior tangents can be drawn. If the circumferences intersect each other, only the two exterior tangents can be drawn. If the circles are tangent externally, two exterior tangents can be drawn, but there can be only one interior tangent. If they are tangent internally, only one exterior tangent can be drawn, and there can be no interior tangent. If one circle is wholly within the other, there can be no common tangent.

Ex. 292. If the two interior, or two exterior, common tangents are drawn to two circles, the segments intercepted between the points of contact are equal.

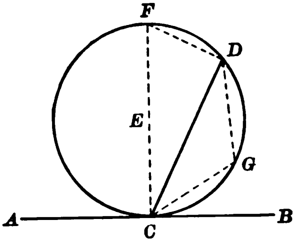
Ex. 293. To construct a right triangle when the hypotenuse and the altitude upon the hypotenuse are given.

Ex. 294. Find the locus of the vertex of a right triangle which has a given hypotenuse as its base.

Ex. 295. Find the locus of the vertex of a triangle which has a given vertical angle and a given base.

Proposition 129. Theorem.

162. *An angle formed by a tangent and a chord is measured by one half the intercepted arc.*



Hypothesis. AB is tangent to the $\odot FDC$ at C, and CD is a chord.

Conclusion. $\angle BCD$ is measured by $\frac{1}{2}$ arc CD, and $\angle ACD$ is measured by $\frac{1}{2}$ arc CFD.

Proof. Draw the diameter CF, and draw FD.

From any point G of the arc CD, draw GC and GD.

$\angle BCF$ is a rt. $\angle$. (?)

$\therefore \angle BCD$ is the complement of $\angle DCF$. (?)

$\angle CDF$ is a rt. $\angle$. (?)

$\therefore \angle F$ is the complement of $\angle DCF$. (?)

$\therefore \angle BCD = \angle F$. (?)

But $\angle F$ is measured by $\frac{1}{2}$ arc CD. (?)

$\therefore \angle BCD$ is measured by $\frac{1}{2}$ arc CD. (?)

$\angle ACD$ is the supplement of $\angle BCD$. (?)

$\angle G$ is the supplement of $\angle F$. (?)

$\therefore \angle ACD = \angle G$. (?)

But $\angle G$ is measured by $\frac{1}{2}$ arc CFD. (?)

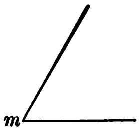
$\therefore \angle ACD$ is measured by $\frac{1}{2}$ arc CFD. (?)

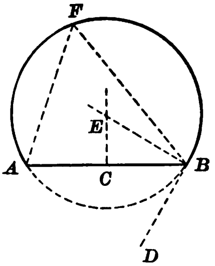
Q.E.D.

Cor. *An angle formed by a tangent and a chord is equal to the angle inscribed in the opposite segment.*

Proposition 130. Problem.

163. ***Upon a given straight line, to describe a segment of a circle which shall contain a given angle.***





To describe a segment on AB such that any $\angle$ inscribed in it shall be equal to $\angle m$.

Construction. Draw CE, the $\perp$ bisector of AB.

At B construct $\angle ABD = \angle m$, and draw $BE \perp$ to BD, meeting CE at E.

With E as a centre, and with EB as a radius, describe a circumference.

Then AFB is the required segment.

Proof. A is a point in the circumference. (?)

$$\angle AFB = \angle ABD = \angle m. \qquad (?)$$

Q.E.F.

Ex. 296. To construct a triangle when the base, the altitude, and the vertical angle are given.

Ex. 297. To construct a triangle when the base, the median, and the vertical angle are given.

Ex. 298. To construct a triangle when the base, a base angle, and the radius of the circumscribed circle are given.

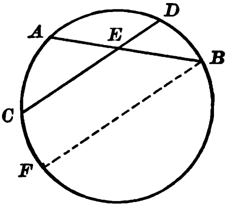
Ex. 299. Prove Prop. 129 by drawing the radius EC and a diameter perpendicular to CD.

Ex. 300. Two tangents AB and AC are drawn to a circle, and D is any point in the major arc BC; prove that the sum of the angles ABD and ACD is constant.

Ex. 301. Two tangents AB and AC are drawn to a circle, and D is the middle point of the arc BC; prove that BD bisects the angle ABC.

Proposition 131. Theorem.

164. *An angle formed by two chords which intersect within a circle is measured by one half the sum of the arcs intercepted by its sides and by the sides of the vertical angle.*



Hypothesis. In the $\odot CDB$, the chords AB and CD intersect at E.

Conclusion. $\angle AEC$ is measured by $\frac{1}{2}$(arc AC + arc DB).

Proof. Draw $BF \parallel$ to DC.

$\angle B$ is measured by $\frac{1}{2}$ arc AF, or $\frac{1}{2}$(arc AC + arc CF). (?)

But arc CF = arc DB. (?)

$\therefore \angle B$ is measured by $\frac{1}{2}$(arc AC + arc DB). (?)

But $\angle B = \angle AEC$. (?)

$\therefore \angle AEC$ is measured by $\frac{1}{2}$(arc AC + arc DB). (?)

Q.E.D.

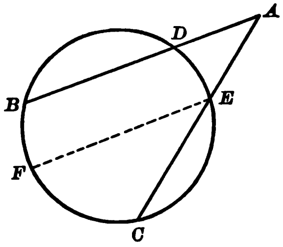
Ex. 302. Two chords, AB and CD, intersect at a point E within the circle whose centre is F; prove that the angle AEC is one half of the sum of the angles AFC and BFD.

Ex. 303. If two chords which intersect within a circle are perpendicular to each other, the sum of the opposite arcs is equal to a semicircumference.

Ex. 304. AB and AC are two chords of a circle, and D and E are the middle points of the subtended arcs; if the chord DE intersects AB and AC at F and G respectively, prove that $AF = AG$.

Proposition 132. Theorem.

165. *An angle formed by two secants which intersect without a circle is measured by one half the difference of the intercepted arcs.*



Hint. Draw $EF \parallel$ to AB.

Proposition 133. Theorem.

166. *An angle formed by a secant and a tangent is measured by one half the difference of the intercepted arcs.*

Proposition 134. Theorem.

167. *An angle formed by two tangents is measured by one half the difference of the intercepted arcs.*

EXERCISES.

305. *AB* and *CD*, two chords of the circle whose centre is *E*, meet when produced at *F*. Prove that the angle *AFC* is equal to one half the difference of the angles *AEC* and *BED*.

306. *AB* is a fixed chord in a circle, and *CD* is another chord of given length. If *AC* and *BD* produced meet at *E*, the angle *E* has the same value whatever the position of *CD*.

307. If a circle is circumscribed about an isosceles triangle, the tangents drawn through the vertices form another isosceles triangle.

308. If two opposite sides of a quadrilateral inscribed in a circle are equal, the other two sides are parallel.

309. State and prove the converse of Ex. 308.

310. The only parallelogram which can be inscribed in a circle is a rectangle.

311. If two chords are drawn perpendicular to a third chord at its extremities, they are equal.

312. If from any point in the circumference of a circle a chord and a tangent are drawn, the perpendiculars drawn to them from the middle point of the subtended arc are equal.

313. *AB* is a chord of a circle whose centre is *C*, and *DE* is the perpendicular drawn to *AB* from any point *D* in the circumference. Prove that the angles *ADE* and *BDC* are equal.

314. If two chords of a circle intersect each other, and make equal angles with the straight line drawn from the point of intersection to the centre of the circle, they are equal.

315. If a hexagon *ABCDEF* is inscribed in a circle, the sum of the angles *A*, *C*, and *E* is equal to the sum of the angles *B*, *D*, and *F*.

316. If two straight lines are drawn through any point of a diagonal of a square parallel to the sides of the square, the points where these lines meet the sides will lie on the circumference of a circle whose centre is the point of intersection of the diagonals.

317. Two circles intersect at *A* and *B*, and through *A* two straight lines *CD* and *EF* are drawn terminated by the circumferences; prove that the angles *CBE* and *DBF* are equal.

318. A secant *ABC* and a tangent *AD* are drawn to a circle from an exterior point *A*, and the bisector of the angle *BDC* meets *AC* at *E*; prove that $AE = AD$.

319. If two circles are tangent to each other, the two tangents drawn from any point in the common tangent passing through the point of contact are equal.

320. *B* is a point in the circumference of a circle whose centre is *C*; *AD*, a tangent at any point *A*, meets *CB* produced at *D*, and *AE* is drawn perpendicular to *CB*. Prove that *AB* bisects the angle *DAE*.

321. *AB* is a chord of a circle, *AC* is a diameter, and *AD* is the perpendicular drawn from *A* to the tangent at *B*; prove that *AB* bisects the angle *CAD*.

322. Two circles are tangent externally, and through the point of contact a straight line is drawn terminated by the circumferences; prove that the radii drawn to its extremities are parallel.

323. Two circles are tangent externally, and through the point of contact a straight line is drawn terminated by the circumferences; prove that the radii at its extremities are parallel.

324. If two straight lines are drawn through the point of contact of two circles which are tangent to each other, the chords of the intercepted arcs are parallel.

325. If two circles are tangent externally, and if two straight lines are drawn through the point of contact, two of the lines which join the middle points of the chords thus formed are parallel.

326. If two circles are tangent internally, and the radius of one circle is the diameter of the other, any chord of the larger drawn from the point of contact is bisected by the circumference of the smaller.

327. Two circles whose centres are *A* and *B* are tangent internally, and a straight line is drawn through the point of contact intersecting the circumferences at *C* and *D*; prove that the radii *AC* and *BD* are parallel.

328. Two sides and the included angle of a quadrilateral are fixed. If the angle opposite the included angle is its supplement, find the locus of the vertex of the angle opposite the given angle.

329. If a quadrilateral is circumscribed about a circle, the angles subtended at the centre by any two opposite sides are supplementary.

330. In a circumscribed trapezoid, the median is equal to one fourth the perimeter.

331. ACB is a central angle, and the secant BE intersects the circumference of the circle at D and AC produced at E; if ED is equal to the radius of the circle, prove that the angle E is equal to one third the angle ACB.

332. AB and CD are two equal chords which intersect at E; prove that $AE = ED$ and $BE = EC$.

333. A circle is inscribed in the triangle ABC. If D, E, and F are the points of contact of AB, AC, and BC respectively, prove that the angle EFD is one half the supplement of the angle A.

334. In any right triangle, the sum of the hypotenuse and the diameter of the inscribed circle is equal to the sum of the legs.

335. If a circle is described on one of the legs of an isosceles triangle as a diameter, the circumference bisects the base of the triangle.

336. In the circle whose centre is E, AB and CD are two diameters perpendicular to each other. Equal distances EF and EG are taken on EA and ED respectively. Prove that BG produced is perpendicular to DF, and that these two straight lines when produced intercept an arc equal to one fourth of the circumference.

337. If two equal circles intersect, and if through one of the points of intersection a straight line is drawn terminated by the circumferences, the straight lines drawn from its extremities to the other point of intersection are equal.

338. Two circles intersect at A and B, and through A any straight line CD is drawn terminated by the circumferences; prove that the angle CBD has the same value whatever the position of CD.

339. If the sum of two opposite angles of a quadrilateral is equal to two right angles, the four vertices lie on the circumference of a circle.

340. Two tangents are drawn to a circle at the opposite extremities of a diameter, and cut off from a third tangent a segment AB; if C is the centre of the circle, prove that ACB is a right angle.

341. To describe a circle touching a given straight line at a given point so that tangents drawn to it from two fixed points in the given line shall be parallel.

342. In the diameter of a circle produced, determine a point such that the tangent drawn from it shall be equal to a given line.

343. In the triangle ABC, if AD and BE are drawn perpendicular to BC and AC respectively, and DE is drawn, prove that the angles ABE and ADE are equal. [HINT. Describe a circle on AB as a diameter.]

344. If from the extremities of the base of a triangle perpendiculars are drawn to the opposite sides, the straight line joining the feet of these perpendiculars will be bisected by the perpendicular drawn to it from the middle point of the base.

345. The perimeter of a circumscribed equilateral triangle is equal to twice the perimeter of the inscribed equilateral triangle.

346. The radius of a circle circumscribed about an equilateral triangle is equal to two thirds the altitude of the triangle.

347. The radius of a circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.

348. To construct an equilateral triangle when the radius of the circumscribed circle is given.

349. To construct an equilateral triangle when the radius of the inscribed circle is given.

350. The vertical angle of an isosceles triangle is 120°. Prove that the radius of the circumscribed circle is equal to a leg of the triangle.

351. If one leg of a right triangle is equal to one half the hypotenuse, the two acute angles of the triangle are 60° and 30° respectively.

352. To describe three circles tangent to each other, the positions of their centres being given.

353. In a right triangle, if a circle is described on one leg as a diameter, the tangent at the point where it cuts the hypotenuse bisects the other leg.

354. Two circles are tangent externally at A, and a common exterior tangent touches them at B and C respectively; prove that the angle BAC is a right angle.

355. Two circles are tangent externally at A, and a common exterior tangent touches them at B and C respectively; prove that the circle described on BC as a diameter passes through A and touches the line joining the centres of the given circles.

356. Two circles intersect at A and B, and through A chords are drawn in each circle tangent to the other circle. If BC and BD are the lines drawn from B to the extremities of these chords, prove that the angle CBD is bisected by AB (produced if necessary).

357. AB is the common chord of two intersecting circles; through C, any point of one circumference, straight lines CAD and CBE are drawn terminated by the other circumference; prove that the length of the chord DE is the same whatever the position of the point C.

358. Through a point A in the circumference of a circle two straight lines BAC and DAE are drawn meeting the circumference at C and E; prove that the bisector of the angles BAE and DAC intersects the circumference at a point equidistant from C and E.

359. AB and AC are two chords of a circle making equal angles with the chord AD; AE and AF are two other chords making equal angles with AD. Prove that BC is parallel to EF.

360. Through a point A in the circumference of a circle two chords AB and AC are drawn; on these as diameters circumferences are described. Prove that the point of intersection of these circumferences lies in the straight line BC.

361. In a right triangle, the straight line drawn from the vertex of the right angle to the centre of the square on the hypotenuse bisects the right angle. [HINT. Describe a circle on the hypotenuse as a diameter.]

362. The tangents to a circle drawn through the vertices of an inscribed rectangle enclose either a rhombus or a square.

363. If ABC is an equilateral triangle inscribed in a circle, and D is any point in the arc BC, prove that $DA = DB + DC$. [HINT. On DA take $DE = DC$, and draw EC.]

364. AB and AC are the tangents drawn to a circle from the exterior point A, and D is any point in the arc BC. If a tangent at D meets AB and AC at E and F respectively, prove that the perimeter of the triangle AEF is equal to $AB + AC$. If G is the centre of the circle, prove that the angle EGF has the same value whatever the position of the point D.

365. To construct a circle which shall have a given radius and touch two given intersecting straight lines.

366. To construct a circle which shall have a given radius and touch a given straight line and a given circle.

367. To construct a circle which shall have a given radius and touch two given circles.

368. To construct a circle which shall have a given radius and touch a given straight line, and whose circumference shall pass through a given point.

369. To construct a circle which shall have a given radius and touch a given circle, and whose circumference shall pass through a given point.

370. To construct a circle which shall touch two given parallel lines and whose circumference shall pass through a given point.

371. To construct a circle which shall touch a given straight line at a given point and whose circumference shall pass through another given point.

372. To construct a circle which shall touch a given circle at a given point and whose circumference shall pass through another given point.

373. To construct a circle which shall touch two given straight lines, one of them at a given point.

374. To construct a circle which shall touch a given circle at a given point and also touch a given straight line.

375. To construct a circle which shall touch a given straight line at a given point and also touch a given circle.

376. The circumference of a circle A passes through the centre of a smaller circle B; prove that their common tangents touch A in points which lie in a straight line tangent to B.

377. Two circles intersect at A and B, and C is a point in the circumference of one of them; the chords CA and CB are produced to meet the other circumference at D and E. Prove that DE is parallel to the tangent at C.

378. Through each of the points of intersection of two circles straight lines are drawn, meeting the circumferences at A and B and C and D respectively. Prove that AC is parallel to BD.

379. If two circles intersect, any two parallel lines drawn through the points of intersection and terminating in the circumferences are equal.

380. Two equal circles are tangent externally, and through the point of contact two chords are drawn, one in each circle, perpendicular to each other; prove that the line joining the other extremities of these chords is equal and parallel to the line joining the centres of the circles.

381. Find the locus of the point of contact of a tangent drawn from a given point to a circle whose centre is a given point.

382. A and B are two given points on the circumference of a given circle, and CD is any diameter. Find the locus of the intersection of CA and DB.

383. Two circles are tangent internally at A, and a chord BC of the larger circle is tangent to the smaller circle at D; prove that AD bisects the angle BAC.

384. On the side AB of a triangle ABC as a diameter a circle is described, and DE is a diameter parallel to BC; prove that BE and BD bisect the interior and exterior angles at B.

385. If any number of triangles have the same base and equal vertical angles, the bisectors of these angles intersect at a common point.

386. A straight line moves so that its extremities lie constantly on two lines which are perpendicular to each other; find the locus of the middle point.

387. Find the locus of the middle point of a chord drawn from a given point in the circumference of a given circle.

388. A chord when produced passes through a given point without a given circle; find the locus of the middle point of the chord.

389. A secant passes through a given point without a given circle; find the locus of the middle point of the entire secant.

390. A secant passes through a given point without a given circle; find the locus of the middle point of the external segment.

391. $ABCD$ is an inscribed quadrilateral. If the sides AB and DC when produced meet at E, and the sides AD and BC when produced meet at F, prove that the bisectors of the angles E and F are perpendicular to each other.

392. Through A, one of the points of intersection of two circles, two straight lines BC and DE are drawn terminating in the circumferences; if BC is fixed in position, prove that the angle formed by producing BD and EC has the same value whatever the position of DE.

393. AB is the common chord of two intersecting circles, and through A the straight line CD is drawn perpendicular to AB and terminating in the circumferences; if CB and DB meet the circumferences again at E and F, prove that AB bisects the angle EAF.

394. Through one of the points of intersection of two circles, each of whose circumferences passes through the centre of the other, a line is drawn intersecting the circumferences in two other points. Prove that the three lines joining these points and the other point of intersection form an equilateral triangle.

395. AB is a chord of a circle, and C is any point in the circumference; CD is drawn perpendicular to AB, and AE is drawn perpendicular to the tangent at C. Prove that DE is parallel to BC.

396. If in any triangle the feet of the perpendiculars drawn from the vertices to the opposite sides are joined by straight lines, the three angles of the triangle thus formed will be bisected by the perpendiculars.

PROBLEMS OF COMPUTATION.

1. Two secants meet without a circle and intercept arcs which are respectively $\frac{5}{9}$ and $\frac{4}{15}$ of the circumference; find the number of degrees in the angle formed by the secants.

2. Two secants which meet without a circle form an angle of 80°. If one of the intercepted arcs is 170°, what is the other?

3. The arc intercepted between two tangents drawn from an exterior point is equal to one eighth of the circumference; find the angle formed by the tangents.

4. Two tangents drawn to a circle form an angle of 80°; find the number of degrees in the arc between the two points of contact.

5. If the angle formed by two tangents is 80°, what is the angle formed by one of the tangents and the chord joining the two points of contact?

6. Two chords of a circle which intersect within the circumference divide the latter into four parts whose lengths taken in order are 89°, 43°, 117°, and 111°; find the angles which the chords make with each other.

7. The bisectors of the angles A, B, and C of an inscribed triangle meet the circumference at D, E, and F respectively. If the angle A is equal to 110°, and the angle B is equal to 30°, find the number of degrees in the angle EDF.

8. Two tangents drawn from an exterior point are perpendicular to each other. If the radius of the circle is 10 inches, find the length of each tangent.

9. On a given line four inches long a segment of a circle is described which contains an angle of 30°. Find the length of the diameter of the circle.

10. A, B, C, and D are four points lying in order on a certain circumference. The arcs AB, BC, and CD have the values 86°, 43°, and 108° respectively. Find the angle formed by the intersection of the chords AC and BD, and the angle formed by the intersection of the chords AB and CD produced.

11. The diagonals AC and BD of an inscribed quadrilateral intersect at E. If the sides AB and BC subtend arcs of 35° and 103° respectively, and the angle between the diagonals is 119°, how many degrees are there in each angle of the quadrilateral?

12. The vertices of a quadrilateral inscribed in a circle divide the circumference into arcs which are to each other as 1, 2, 3, and 4. Find the angles between the opposite sides of the quadrilateral.

BOOK IV.

SIMILAR POLYGONS.

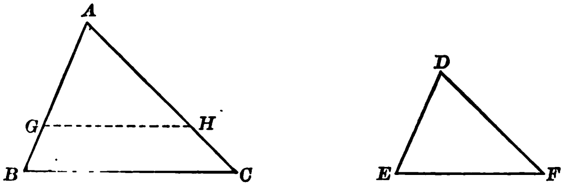
168. Similar figures have been defined in § 3 as figures which have the same shape. In order that two polygons may have the same shape, two things are essential — equality of angles and proportionality of sides. Hence, two polygons are *similar* when their homologous angles are equal and their homologous sides are proportional.

In similar triangles, the homologous sides lie opposite the equal angles. In similar polygons of any kind, the homologous sides are those which join the vertices of equal angles in the respective polygons.

In two similar polygons, the ratio of any two homologous sides is called the *ratio of similitude* of the polygons.

Proposition 135. Theorem.

169. *Two triangles are similar when they are mutually equiangular.*



Hypothesis. In the $\triangle ABC$ and DEF, $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$.

Conclusion. $\triangle$s ABC and DEF are similar.

Proof. On AB lay off $AG = DE$, and on AC lay off $AH = DF$; also draw GH.

Then $\triangle AGH = \triangle DEF$. (?)

$\therefore \angle AGH = \angle E$. (?)

But $\angle E = \angle B$. (?)

$\therefore \angle AGH = \angle B$. (?)

$\therefore GH$ is $\parallel$ to BC. (?)

$\therefore \dfrac{AB}{AG} = \dfrac{AC}{AH}$. (?)

$\therefore \dfrac{AB}{DE} = \dfrac{AC}{DF}$. (?)

In like manner it may be proved that $\dfrac{AB}{DE} = \dfrac{BC}{EF}$.

$\therefore$ $\triangle$s ABC and DEF are similar. (?)

Q.E.D.

Cor. I. *Two triangles are similar when two angles of one are equal respectively to two angles of the other.*

Cor. II. *Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.*

Cor. III. *If a straight line is drawn through two sides of a triangle parallel to the third side, the triangle thus formed is similar to the whole triangle.*

Ex. 397. Two isosceles triangles are similar when their vertical angles are equal.

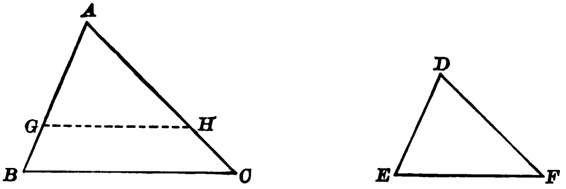
Ex. 398. Any two altitudes of a triangle are inversely proportional to the corresponding bases.

Ex. 399. In the triangle ABC, the altitudes AD and BE intersect at F; prove that $AF : EF :: BF : DF$.

Ex. 400. From two points, A and B, on an arc of a circle, straight lines AC, AD, BC, and BD are drawn to the extremities of the arc; if AD and BC intersect at E, prove that the triangles AEC and BED are similar.

Proposition 136. Theorem.

170. *Two triangles are similar when their homologous sides are proportional.*



Hypothesis. In the $\triangle$s ABC and DEF, $\frac{AB}{DE}=\frac{AC}{DF}=\frac{BC}{EF}$.

Conclusion. $\triangle$s ABC and DEF are similar.

Proof. On AB lay off $AG=DE$, and on AC lay off $AH=DF$; also draw GH.

$$\frac{AB}{AG}=\frac{AC}{AH}. \qquad (?)$$

$$\therefore\ GH \text{ is } \parallel \text{ to } BC. \qquad (?)$$

$$\therefore\ \triangle\text{s } AGH \text{ and } ABC \text{ are similar.} \qquad (?)$$

$$\therefore\ \frac{AB}{AG}=\frac{BC}{GH}. \qquad (?)$$

$$\text{But } \frac{AB}{DE}=\frac{BC}{EF}. \qquad (?)$$

$$\therefore\ GH=EF. \qquad (?)$$

$$\therefore\ \triangle AGH=\triangle DEF. \qquad (?)$$

$$\therefore\ \triangle\text{s } ABC \text{ and } DEF \text{ are similar.} \qquad (?)$$

Q.E.D.

Ex. 401. *AB* and *AC* are two chords drawn from any point *A* in the circumference of the circle *ABC*, and *AD* is a diameter; *AB* and *AC* produced meet the tangent drawn through *D* at *E* and *F* respectively. Prove that the triangles *ABC* and *AEF* are similar.

Proposition 137. Theorem.

171. *Two triangles are similar when an angle of one is equal to an angle of the other and the sides including these angles are proportional.*

HINT. Use the same figure as in Prop. 135.

Ex. 402. If AD and BE are two altitudes of the triangle ABC, prove that the triangle CDE is similar to the triangle ABC.

Ex. 403. If the sides AB and DC of an inscribed quadrilateral are produced to meet at E, prove that the triangles ADE and BCE are similar.

Ex. 404. In the right triangle ABC, DE is drawn from a point D in AC perpendicular to the hypotenuse AB; prove that $AB \times AE = AC \times AD$.

Ex. 405. If AB is a chord of a circle, and CE is any chord drawn through C, the middle point of the arc AB, cutting the chord AB at D, prove that the chord AC is a mean proportional between CD and CE.

Ex. 406. If two circles are tangent to each other, the chords formed by a straight line drawn through the point of contact are proportional to the diameters.

Ex. 407. If two circles are tangent to each other, secants drawn through the point of contact terminating in the circumferences are divided proportionally at the point of contact.

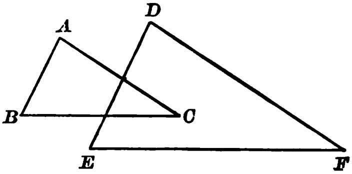
Ex. 408. AB is any chord, and AC is tangent to the circle at A; CDE is a line parallel to AB cutting the circumference at D and E. Prove that the triangles ACD and ABE are similar.

Ex. 409. The diagonals AC and BD of the quadrilateral $ABCD$ intersect at E. If $AE : EC :: BE : ED$, prove that the triangles AEB and CED are similar.

Ex. 410. A is any point in the circumference of the circle ABC; with A as a centre and with any radius, a circle is described whose circumference cuts the given circumference at B and D; from A any line is drawn cutting the chord BD at E and the circumferences BFD and ABC at F and C respectively. Prove that AF is a mean proportional between AE and AC.

Proposition 138. Theorem.

172. *Two triangles are similar when their sides are respectively parallel to each other.*



Hypothesis. In the $\triangle$s ABC and DEF, AB is $\parallel$ to DE, AC is $\parallel$ to DF, and BC is $\parallel$ to EF.

Conclusion. $\triangle$s ABC and DEF are similar.

Proof. There are three pairs of homologous $\angle$s, A and D, B and E, and C and F.

In any pair of homologous $\angle$s, the $\angle$s are either equal or supplementary. (?)

Hence one of the following statements must be true: —

(i) In every pair of homologous $\angle$s, the $\angle$s are supplementary.

(ii) In two pairs of homologous $\angle$s, the $\angle$s are supplementary, and in the third pair, they are equal.

(iii) In one pair of homologous $\angle$s, the $\angle$s are supplementary, and in the other two pairs, they are equal.

(iv) In every pair of homologous $\angle$s, the $\angle$s are equal.

The first two statements cannot be true. (?)

Furthermore, the third statement cannot be true. (?)

$\therefore$ the fourth statement is true, and the $\triangle$s are similar. (?)

Q.E.D.

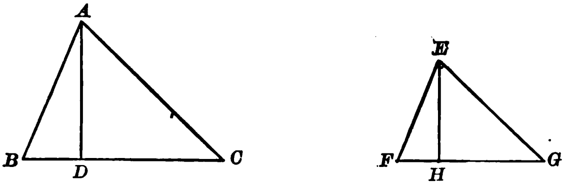
Proposition 139. Theorem.

173. *Two triangles are similar when their sides are respectively perpendicular to each other.*

The method of proof is the same as that of Prop. 138.

Proposition 140. Theorem.

174. *If two triangles are similar, their homologous altitudes are in the same ratio as any two homologous sides.*



Hypothesis. AD and EH are homologous altitudes of the similar △ ABC and EFG.

Conclusion. $\frac{AD}{EH}=\frac{AB}{EF}=\frac{AC}{EG}=\frac{BC}{FG}.$

HINT. Prove that △ ABD and EFH are similar.

Ex. 411. AB is a diameter of a circle, and AD is a secant which cuts the circumference at C and the tangent drawn through B at D; prove that the product $AC \times AD$ has the same value whatever the position of AD.

Ex. 412. AB and CD are two diameters of a circle which are perpendicular to each other, and CE and CF are two chords drawn intersecting AB at G and H respectively; prove that $CE \times CG = CF \times CH$.

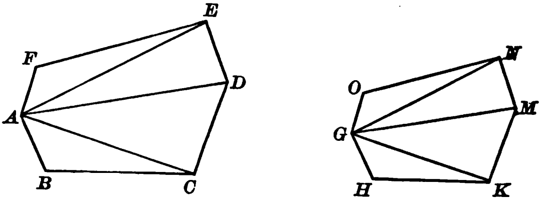
Ex. 413. In the triangle ABC, the bisector of the vertical angle BAC intersects the base at D and the circumference of the circumscribed circle at E; prove that $AB \times AC = AD \times AE$.

Ex. 414. A is the vertex of an isosceles triangle ABC inscribed in a circle, and AE is a chord of the circle intersecting the base of the triangle at D; prove that $\overline{AB}^2 = AD \times AE$.

Ex. 415. D is any point in the diameter AB of a circle whose centre is C; DE is drawn perpendicular to AB, meeting the circumference at E, and DF is the perpendicular drawn from D to the radius CE. Prove that DE is a mean proportional between AC and EF.

Proposition 141. Theorem.

175. *If two polygons are similar, the diagonals drawn from homologous vertices divide them into the same number of triangles, similar each to each, and similarly placed.*



Hypothesis. *ABCDEF* and *GHKMNO* are similar polygons, and from the homologous vertices are drawn the diagonals *AC*, *AD*, *AE*, *GK*, *GM*, and *GN*.

Conclusion. $\triangle s$ *ABC* and *GHK* are similar, $\triangle s$ *ACD* and *GKM* are similar, and so on.

Proof. $\angle B = \angle H$, and $\frac{AB}{GH} = \frac{BC}{HK}$. (?)

$\therefore \triangle s$ *ABC* and *GHK* are similar. (?)

$\angle BCD = \angle HKM$. (?)

$\angle BCA = \angle HKG$. (?)

$\therefore \angle ACD = \angle GKM$. (?)

$\frac{BC}{HK} = \frac{CD}{KM}$. (?)

$\frac{BC}{HK} = \frac{AC}{GK}$. (?)

$\therefore \frac{CD}{KM} = \frac{AC}{GK}$. (?)

$\therefore \triangle s$ *ACD* and *GKM* are similar. (?)

In like manner it may be proved that every other $\triangle$ of *ABCDEF* is similar to the triangle similarly situated in *GHKMNO*. Q.E.D.

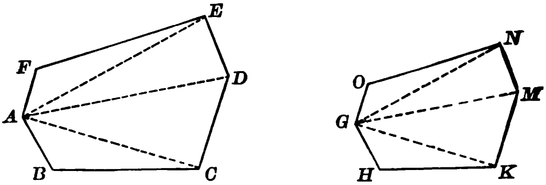
Proposition 142. Theorem.

176. *Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.*

Since the homologous ∠s are either homologous ∠s of similar △s, or else the sums of such ∠s, the polygons may be proved mutually equiangular. Since any two homologous sides are proportional to the adjacent homologous diagonals, the homologous sides may be proved proportional.

Proposition 143. Problem.

177. *To construct a polygon similar to a given polygon upon a given line homologous to a given side of the polygon.*



To construct a polygon similar to *ABCDEF* upon *GH* homologous to *AB*.

Construction. Draw the diagonals *AC*, *AD*, and *AE*.

Draw *GK* and *HK*, making ∠s *HGK* and *GHK* equal respectively to ∠s *BAC* and *ABC*.

[To be completed by the student.]

Proposition 144. Theorem.

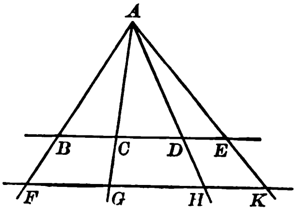
178. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*

Write the continued proportion formed by the homologous sides of the polygons, and consult § 100.

Proposition 145. Theorem.

179. *If two parallel lines are cut by three or more transversals which intersect at a common point, the corresponding segments of the parallel lines are proportional.*

CASE I. *When the parallel lines are on the same side of the common point.*



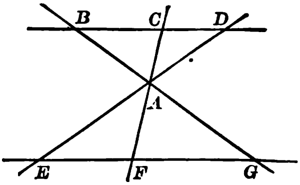
Hypothesis. The transversals AF, AG, AH, and AK, which intersect at A, cut the ∥ lines BE and FK at the points B, C, D, and E, and F, G, H, and K, respectively.

Conclusion. $\frac{BC}{FG}=\frac{CD}{GH}=\frac{DE}{HK}$.

HINT. Prove that △ ABC and AFG are similar, △ ACD and AGH are similar, and △ ADE and AHK are similar.

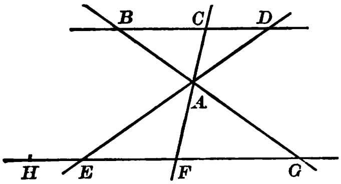
CASE II. *When the parallel lines are on opposite sides of the common point.*

$\frac{BC}{FG}=\frac{CD}{EF}$ is the conclusion.



Proposition 146. Theorem.

180. *If three or more lines which are not parallel intercept proportional segments on two parallel lines, they intersect at a common point.*



Hypothesis. The lines BG, CF, and DE, which are not ∥, cut the ∥ lines BD and EG so that $\frac{BC}{FG}=\frac{CD}{EF}$.

Conclusion. BG, CF, and DE intersect at a common point.

Proof. Let BG and CF intersect at A.

Draw DA, and suppose it to cut EG at H when produced.

$$\text{Then } \frac{BC}{FG}=\frac{CD}{HF}. \qquad (?)$$

$$\text{But } \frac{BC}{FG}=\frac{CD}{EF}. \qquad (?)$$

$$\therefore HF=EF. \qquad (?)$$

$$\therefore H \text{ coincides with } E. \qquad (?)$$

$\therefore DA$ produced passes through E, and BG, CF, and DE intersect at the common point A.

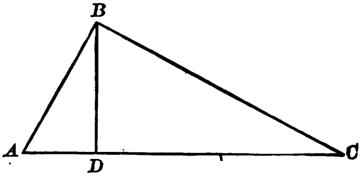
Q.E.D.

Ex. 416. D is the middle point of the base BC of the triangle ABC; if any line parallel to BC cuts AB, AD, and AC at E, F, and G respectively, prove that $EF=FG$.

Ex. 417. The diagonals of a trapezoid divide each other into segments which are proportional.

Proposition 147. Theorem.

181. *In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the perpendicular is a mean proportional between the segments of the hypotenuse, and either leg is a mean proportional between the whole hypotenuse and the adjacent segment.*



Hypothesis. In the rt. △ ABC, BD is drawn from the vertex of the rt. ∠ ABC ⊥ to the hypotenuse AC.

Conclusion. $AD : BD :: BD : DC$,

$AC : AB :: AB : AD$,

and $AC : BC :: BC : DC$.

Proof. ⊿ ABD and ABC are similar. (?)

⊿ BDC and ABC are similar. (?)

∴ ⊿ ABD and BDC are similar. (?)

∴ $AD : BD :: BD : DC$,

$AC : AB :: AB : AD$,

and $AC : BC :: BC : DC$. (?)

Q.E.D.

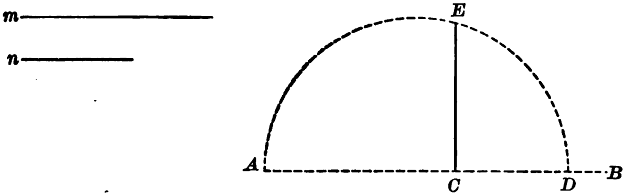
Cor. I. *In a right triangle, the perpendicular drawn from the vertex of the right angle to the hypotenuse divides the triangle into two triangles which are similar to the whole triangle, and also to each other.*

Cor. II. *If from any point in the circumference of a circle a perpendicular is drawn to the diameter, the perpendicular is a mean proportional between the segments of the diameter, and the chord joining the point to either extremity of the diameter is a mean proportional between the whole diameter and the adjacent segment.*

Ex. 418. If BD is the perpendicular drawn from the vertex of the right angle to the hypotenuse of the right triangle ABC, prove that $AB \times BC = AC \times BD$.

Proposition 148. Problem.

182. *To find a mean proportional between two given straight lines.*



To find a mean proportional between m and n.

Construction. On the indefinite line AB, lay off $AC = m$, and $CD = n$.

On AD as a diameter, describe a semi-circumference.

At C erect $CE \perp$ to AD, meeting the circumference at E.

Then CE is the required mean proportional.

[The proof is left to the student.]

Scholium. This construction may be used to construct a line whose numerical measure is a surd. For example, $\sqrt{6} = \sqrt{3 \times 2}$. Then by taking AC equal to 3 units and CD equal to 2 units, the value of CE is $\sqrt{6}$.

Proposition 149. Theorem.

183. *In a right triangle, if a perpendicular is drawn from the vertex of the right angle to the hypotenuse, the squares of the legs are in the same ratio as the adjacent segments of the hypotenuse, and the ratio of the square of the hypotenuse to the square of either leg is equal to the ratio of the whole hypotenuse to the adjacent segment.*

To prove the first part, apply § 92 to both results obtained in the latter part of Prop. 147, and divide one result by the other.

Proposition 150. Theorem.

184. *In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*

Apply § 97 to the proportion obtained in the first part of Prop. 149, and compare this result with the proportion obtained in the second part of Prop. 149.

Ex. 419. If a perpendicular is drawn from the vertex of a triangle to the base, the difference of the squares of the sides including the vertical angle is equal to the difference of the squares of the segments of the base.

Ex. 420. If the perpendiculars from the vertices of the triangle ABC to the opposite sides intersect at D, prove that $\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2$.

Ex. 421. AB is the hypotenuse of the right triangle ABC, and D is the middle point of the leg AC; prove that $\overline{AB}^2 - \overline{BD}^2 = 3\,\overline{CD}^2$.

Ex. 422. In the right triangle ABC, D is the middle point of the leg AC, and DE is drawn perpendicular to the hypotenuse AB; prove that $\overline{BE}^2 - \overline{AE}^2 = \overline{BC}^2$.

Ex. 423. In the right triangle ABC, D and E are points in the legs AC and BC respectively; prove that $\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2$.

Ex. 424. If the acute angle A of the right triangle ABC is double the acute angle B, prove that $\overline{AC}^2 = \frac{1}{3}\,\overline{BC}^2$.

Ex. 425. If AD is the altitude of the equilateral triangle ABC, prove that $4\,\overline{AD}^2 = 3\,\overline{BC}^2$.

Ex. 426. In the triangle ABC, if DE, DF, and DG are the perpendiculars drawn from any point D to the sides AB, AC, and BC respectively, prove that $\overline{AE}^2 + \overline{BG}^2 + \overline{CF}^2 = \overline{BE}^2 + \overline{CG}^2 + \overline{AF}^2$.

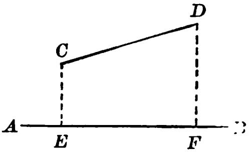
Ex. 427. If AD and BE are the medians drawn from the vertices of the acute angles of the right triangle ABC, prove that $4\,\overline{AD}^2 + 4\,\overline{BE}^2 = 5\,\overline{AB}^2$.

Ex. 428. The diagonals of a quadrilateral are perpendicular to each other; prove that the sum of the squares of one pair of opposite sides is equal to the sum of the squares of the other pair.

Ex. 429. The legs of a right triangle are a and b, and the perpendicular drawn from the vertex of the right angle to the hypotenuse is p; prove that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

PROJECTION.

185. The *projection of a point* upon a straight line is the foot of the perpendicular drawn from the point to the line. For example, the projection of the point C upon the line AB is the point E. In case a point lies in the line, the point is the projection of itself.



The *projection of a straight line* upon another straight line is the part of the latter included between the projections of the extremities of the former. For example, the projection of the line CD upon the line AB is EF.

Ex. 430. The projections of a straight line upon two parallel lines are equal.

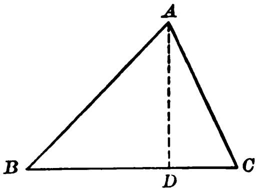
Ex. 431. AB and CD are respectively the projections of the straight line EF on two straight lines which are perpendicular to each other; prove that $\overline{AB}^2 + \overline{CD}^2 = \overline{EF}^2$.

Proposition 151. Theorem.

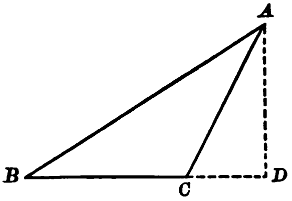
186. *In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by twice the product of one of these sides and the projection of the other side upon it.*

Case I. *When the projection of the vertex upon the base is in the base.*

Case II. *When the projection of the vertex upon the base is in the base produced.*



Case I.



Case II.

Hypothesis. In either case, B is an acute $\angle$ of the $\triangle\, ABC$, and BD is the projection of the side AB upon the side BC.

Conclusion. $\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 - 2\,BC \times BD.$

Proof. In Case I., $CD = BC - BD.$

In Case II., $CD = BD - BC.$

In either case, $\overline{CD}^2 = \overline{BC}^2 + \overline{BD}^2 - 2\,BC \times BD.$ (?)

Then $\overline{AD}^2 + \overline{CD}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{BD}^2 - 2\,BC \times BD.$ (?)

$\therefore\ \overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2 - 2\,BC \times BD.$ (?)

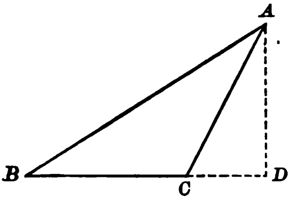
Q.E.D.

Ex. 432. ABC and ADC are two angles inscribed in a semicircle, and AE and CF are perpendiculars drawn to BD produced from A and C, respectively; prove that $\overline{BE}^2 + \overline{BF}^2 = \overline{DE}^2 + \overline{DF}^2$.

Proposition 152. Theorem.

187. *In an obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, increased by twice the product of one of these sides and the projection of the other side upon it.*

HINT. Take $BD = BC + CD$, and then proceed as in Prop. 151.



Ex. 433. If one of the angles of a triangle is equal to 60°, the square of the side opposite this angle is equal to the sum of the squares of the other two sides, diminished by the product of these two sides.

Ex. 434. If one of the angles of a triangle is equal to 120°, the square of the side opposite this angle is equal to the sum of the squares of the other two sides, increased by the product of these two sides.

Ex. 435. In an isosceles triangle, the square of a leg is equal to the square of any straight line drawn from the vertex to the base, increased by the product of the segments of the base.

Ex. 436. The square of the base of an isosceles triangle is equal to twice the product of either leg and the projection of the base upon that leg.

Ex. 437. ABC is an isosceles triangle having each of the base angles double the vertical angle A, and the bisector of the angle ABC meets AC at D; prove that BC is a mean proportional between AC and CD.

Ex. 438. The perpendiculars drawn from two vertices of a triangle to the opposite sides divide each other into segments which are reciprocally proportional.

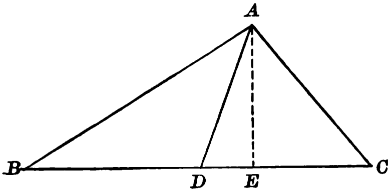
Ex. 439. The squares of two chords drawn from the same point in a circumference are in the same ratio as their projections upon the diameter drawn from the same point.

Proposition 153. Theorem.

188. *In any triangle, if a median is drawn,*

(i) *the sum of the squares of the sides including the vertical angle is equal to twice the square of half the base increased by twice the square of the median;*

(ii) *the difference of the squares of the sides including the vertical angle is equal to twice the product of the base and the projection of the median upon the base.*



Hypothesis. In the $\triangle ABC$, in which $AB > AC$, AD is the median, and DE is the projection of AD upon the base BC.

Conclusion. (i) $\overline{AB}^2 + \overline{AC}^2 = 2\,\overline{BD}^2 + 2\,\overline{AD}^2$.

(ii) $\overline{AB}^2 - \overline{AC}^2 = 2\,BC \times DE$.

Proof. $\angle ADB$ is obtuse. (?)

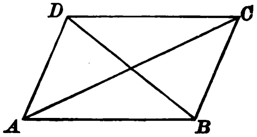
$\angle ADC$ is acute. (?)

Then consult Prop. 151 and Prop. 152, and find the sum and the difference of the results thus obtained.

Ex. 440. The base BC of the triangle ABC is a given line, the middle point of which is the centre of a given circle; the vertex A is a point on the circumference of the circle. Prove that $\overline{AB}^2 + \overline{AC}^2$ is constant whatever the position of A.

Proposition 154. Theorem.

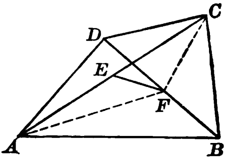
189. *In any parallelogram, the sum of the squares of the four sides is equal to the sum of the squares of the diagonals.*



HINT. Apply Prop. 153 (i) to ⊿s ABC and ADC, and add the results thus obtained.

Proposition 155. Theorem.

190. *In any trapezoid or trapezium, the sum of the squares of the four sides is equal to the sum of the squares of the diagonals increased by four times the square of the line joining the middle points of the diagonals.*



HINT. Draw AF and CF, and use the same method of proof as in Prop. 154.

Ex. 441. Two circles intersect at A and B, and at A tangents are drawn, one in each circle, meeting the circumferences at C and D; prove that AB is a mean proportional between BC and BD.

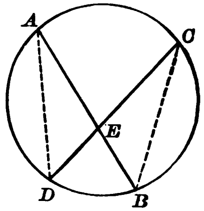
Ex. 442. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the circumference of the smaller circle.

Ex. 443. From the point A, the tangents AB and AC are drawn to the circle whose centre is D, and CE is drawn perpendicular to BD produced; prove that $AB : BD :: BE : CE$.

Ex. 444. If two circles are tangent externally, the segment of an exterior common tangent which is included between the points of contact is a mean proportional between the diameters of the circles.

Proposition 156. Theorem.

191. *If through a fixed point within a circle any chord is drawn, the product of its two segments is constant whatever the direction of the chord.*



Hypothesis. In the $\odot ACD$, AB and CD are two chords drawn through the point E.

Conclusion. $AE \times EB = CE \times ED$.

Proof. Draw AD and CB.

$\triangle$ AED and CEB are similar. (?)

$\therefore AE : CE :: ED : EB$. (?)

$\therefore AE \times EB = CE \times ED$. (?)

Q.E.D.

COR. *If two chords intersect within a circle, their segments are reciprocally proportional.*

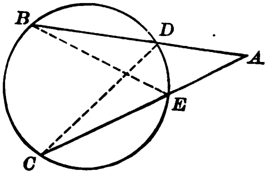
Ex. 445. Either segment of the least chord that can be drawn through a fixed point within a circle is a mean proportional between the segments of any other chord drawn through that point.

Ex. 446. AB and CD are two parallel lines, and E is the middle point of CD; AC and BE intersect at F, and AE and BD intersect at G. Prove that FG is parallel to AB.

Ex. 447. In the circle ABC, A and C are the extremities of a
er, and B is any point on the circumference; the chords AB
produced meet the tangents drawn through A and C at E and
ively. Prove that AC is a mean proportional between AD

Proposition 157. Theorem.

192. *If from a fixed point without a circle any secant is drawn, the product of the whole secant and its external segment is constant whatever the direction of the secant.*

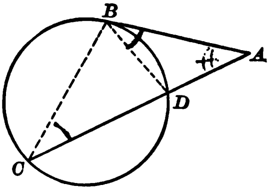


Hint. Draw BE and CD, and prove that $\triangle$ ABE and ACD are similar.

Cor. *If from a point without a circle two secants are drawn, the whole secants and their external segments are reciprocally proportional.*

Proposition 158. Theorem.

193. *If from a point without a circle a tangent and a secant are drawn, the tangent is a mean proportional between the whole secant and its external segment.*



Hint. Draw BC and BD.

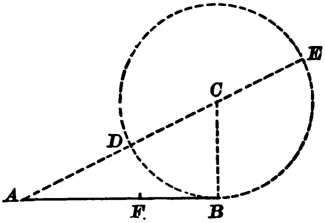
Ex. 448. If two circles intersect, the tangents drawn from any point in their common chord produced are equal.

Ex. 449. If two circles intersect, the segment of a common tangent which is included between the points of contact is bisected by the common chord produced.

Proposition 159. Problem.

194. *To divide a given straight line in extreme and mean ratio.* [See § 108 for definition.]

CASE I. *To divide the line internally.*



To divide the line AB internally in extreme and mean ratio.

Construction. At B erect the $\perp BC$ equal to $\frac{1}{2}AB$.

With C as a centre and with CB as a radius, describe the circumference DEB.

Draw AC, cutting the circumference at D, and produce it to meet the circumference again at E.

On AB lay off $AF = AD$.

Then AB is divided internally in extreme and mean ratio at F.

Proof. $AE : AB :: AB : AD$. (?)

Then $AE - AB : AB :: AB - AD : AD$. (?)

But $AE - AB = AE - DE = AF$. (?)

$AB - AD = FB$. (?)

Hence $AF : AB :: FB : AF$. (?)

$\therefore AB : AF :: AF : FB$. (?)

CASE II. *To divide the line externally.*

Construction. Draw the ⊙ and secant as in Case I.

On BA produced lay off $AG = AE$.

Then AB is divided externally in extreme and mean ratio at G.

Proof. $AE : AB :: AB : AD.$ (?)

Then $AE + AB : AE :: AB + AD : AB.$ (?)

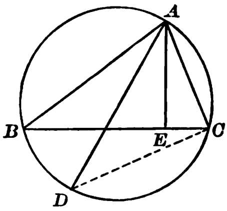
[To be completed by the student.]

Ex. 450. Compute the numerical value of the ratios $\frac{AB}{AF}$ and $\frac{AB}{AG}$ in Prop. 159.

Ex. 451. Given the longer segment of a line divided internally in extreme and mean ratio, to construct the line.

Proposition 160. Theorem.

195. *In any triangle, the product of any two sides is equal to the product of the diameter of the circumscribed circle and the perpendicular drawn to the third side from the vertex of the opposite angle.*



Hypothesis. AD is the diameter of the ⊙ circumscribed about the $\triangle ABC$, and AE is the $\perp$ drawn from A to BC.

Conclusion. $AB \times AC = AD \times AE.$

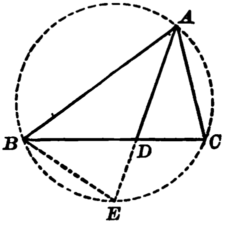
HINT. Draw DC, and prove that $\triangle$s ABE and ADC are similar.

Ex. 452. D is a point in the base BC of the isosceles triangle ABC; prove that the circles circumscribed about the triangles ABD and ACD are equal.

Ex. 453. D is a point in the base BC of the triangle ABC; prove that the diameters of the circles circumscribed about the triangles ABD and ACD are in the ratio of AB to AC.

Proposition 161. Theorem.

196. *In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, increased by the square of the bisector.*



Hypothesis. In the $\triangle ABC$, AD is the bisector of the $\angle BAC$.

Conclusion. $AB \times AC = BD \times DC + \overline{AD}^2$.

Proof. Circumscribe a ⊙ about the $\triangle ABC$.

Produce AD to meet the circumference at E, and draw BE.

$\triangle$s ABE and ADC are similar. (?)

$\therefore AB : AD :: AE : AC$. (?)

$\therefore AB \times AC = AD \times AE$ (?)

$= AD \times (DE + AD)$ (?)

$= AD \times DE + \overline{AD}^2$. (?)

But $AD \times DE = BD \times DC$. (?)

$\therefore AB \times AC = BD \times DC + \overline{AD}^2$. (?)

Q.E.D.

EXERCISES.

454. The chords AB and CD of a circle are produced towards B and C respectively, meeting at the point E, and the line EF is drawn parallel to AD, meeting CB produced at F; prove that EF is a mean proportional between BF and CF.

455. C is the centre of a circle, and A is any point within it; CA is produced to a point B such that the radius is a mean proportional between CA and CB. If D is any point on the circumference, prove that the angles CDA and CBD are equal.

456. If lines are drawn from any point E to the vertices of the rectangle $ABCD$, prove that $\overline{AE}^2 + \overline{CE}^2 = \overline{BE}^2 + \overline{DE}^2$.

457. If D is the middle point of the hypotenuse AB of the right triangle ABC, prove that $\overline{AB}^2 + \overline{AC}^2 + \overline{BC}^2 = 8\,\overline{CD}^2$.

458. ABC and DBC are two right triangles having a common hypotenuse BC; if AB is greater than BD, prove that AC is less than CD.

459. In the triangle ABC, $\overline{AC}^2 - \overline{BC}^2 = \frac{1}{2}\overline{AB}^2$; prove that a perpendicular drawn from C to AB will divide the latter into segments which are to each other as 3 to 1.

460. From the vertex A of a triangle ABC, to draw a line meeting BC produced at D so that AD shall be a mean proportional between BD and CD.

461. Either leg of an isosceles triangle is a mean proportional between half the base and the segment of the base (produced if necessary) which is cut off by a straight line drawn from the vertex perpendicular to the leg.

462. If two chords of a circle are drawn from the point A in the circumference, prove that the chords and the segments cut off from them by a line parallel to the tangent through A are reciprocally proportional.

463. If two tangents are drawn to a circle at the extremities of a diameter, the portion of any third tangent intercepted between them is divided at its point of contact into segments whose product is equal to the square of the radius.

464. AB is the hypotenuse of the right triangle ABC, and AD is the bisector of the angle A; prove that $BC : AC :: AB - AC : CD$.

465. In the equilateral triangle ABC, D is any point in the side BC; E and F are points in AC produced, such that $CE = CB$ and $CF = CD$. If BF and DE intersect at G, prove that $CG : CD :: BC : BC + CD$.

466. To draw through A, a point without a given circle, a secant ABC such that BC is a mean proportional between AB and AC.

467. To draw through A, one of the points of intersection of two circles, a secant BAC such that the two chords thus formed shall be in a given ratio.

468. In the triangle ABC, a perpendicular is drawn from A to the opposite side, meeting it at D between B and C; if AD is a mean proportional between BD and DC, prove that BAC is a right angle.

469. In the triangle ABC, a perpendicular is drawn from A to the opposite side, meeting it at D between B and C; if AB is a mean proportional between BC and BD, prove that BAC is a right angle.

470. Two triangles have the same base, and their vertices lie in a line parallel to the base; through the point of intersection of their sides is drawn a straight line parallel to the base, and terminated by the sides which do not intersect. Prove that the segments of this line are equal.

471. A straight line parallel to the bases of a trapezoid, which passes through the point of intersection of the diagonals and is terminated by the legs, is bisected by the diagonals.

472. AD and BC are the bases of the trapezoid $ABCD$, whose diagonals intersect at E. If F is the middle point of BC, prove that FE produced bisects AD.

473. AD and BC are the bases of the trapezoid $ABCD$, and $AD = \frac{1}{2} BC$; if the diagonals intersect at E, prove that $AE = \frac{1}{2} CE$ and $DE = \frac{1}{2} BE$.

474. Two circles, whose centres are A and B, touch each other externally at C; a common tangent touches the circles at D and E respectively, and meets AB produced at F; prove that $FD \times FE = \overline{FC}^2$.

475. In the right triangle ABC, BD is the perpendicular drawn from the vertex of the right angle B to the hypotenuse; from E, any point in AB, EF is drawn perpendicular to AC; FG is drawn perpendicular to AB. Prove that CE is parallel to DG.

476. In the triangle ABC, the angles B and C are acute; if E and F are the feet of perpendiculars drawn from B and C respectively to the opposite sides, prove that $\overline{BC}^2 = AB \times BF + AC \times CE$.

477. To inscribe in a given circle a triangle similar to a given triangle.

478. To circumscribe about a given circle a triangle similar to a given triangle.

479. In the quadrilateral $ABCD$, the angles A and C are right angles; from E, any point in the diagonal BD, the lines EF and EG are drawn perpendicular to AB and CD respectively. Prove that

$$\frac{EF}{AD} + \frac{EG}{BC} = 1.$$

480. In any quadrilateral, the sum of the squares of the diagonals is double the sum of the squares of the lines joining the middle points of the opposite sides.

481. In any rhombus, the sum of the squares of the diagonals is equal to the square of one half the perimeter.

482. The perpendicular drawn from any point of a circumference to a chord is a mean proportional between the perpendiculars drawn from the same point to the tangents at the extremities of the chord.

483. Two tangents AB and AC are drawn to a circle, and BD is drawn perpendicular to the diameter CE; prove that BD is bisected by the straight line AE.

484. Two circles intersect at A and B, and through A a secant CAD is drawn terminated by the circumferences; prove that BC and BD are in the same ratio as the diameters.

485. The line drawn through the centres of two circles intersects the circumferences at A, B, C, and D, and meets the exterior common tangent at E; prove that $AE \times DE = BE \times CE$.

486. If two chords of a circle are perpendicular to each other, the sum of the squares of the four segments is equal to the square of the diameter.

487. Two circles are tangent externally at A, and through A three secants BAC, DAE, and FAG are drawn terminated by the circumferences; prove that the triangles BDF and CEG are similar.

488. From one vertex A of a parallelogram a straight line is drawn intersecting the diagonal BD at E, the side CD at F, and the side BC produced at G; prove that AE is a mean proportional between EF and EG.

489. In the circle whose diameter is AB, any two chords AC and BD are drawn from the extremities of the diameter intersecting within the circle at E; prove that $\overline{AB}^2 = AE \times AC + BE \times BD$.

490. From one vertex A of a parallelogram a straight line is drawn intersecting the side CD at E and the side BC produced at F; prove that the product $BF \times DE$ has the same value whatever the position of AF.

491. Through E, any point in the diagonal AC of the parallelogram $ABCD$, a line is drawn intersecting AB at F, CB produced at G, CD at H, and AD produced at K; prove that $EF \times EG = EH \times EK$.

492. $ABCD$ is a parallelogram, and E and F are points in a straight line parallel to AB; EA and FB meet at G, and ED and FC meet at H. Prove that GH is parallel to AD.

493. In AB, the diameter of a circle, C and D are two points equally distant from the centre, and E is any point on the circumference; prove that $\overline{EC}^2 + \overline{ED}^2 = \overline{AC}^2 + \overline{AD}^2$.

494. In an inscribed quadrilateral $ABCD$, the diagonals intersect at E; prove that $\frac{AB \times AD}{CB \times CD} = \frac{AE}{CE}$.

495. If the homologous sides of two similar triangles are parallel, the lines which join their homologous vertices intersect at a common point.

496. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the exterior angle at the opposite vertex, diminished by the square of the bisector.

497. In an obtuse triangle, to draw a line from the vertex of the obtuse angle to the opposite side which shall be a mean proportional between the segments of that side.

498. To construct a circle which shall touch a given straight line and whose circumference shall pass through two given points.

499. To construct a circle which shall touch two given straight lines and whose circumference shall pass through a given point.

500. To construct a triangle when the vertical angle, the base, and the ratio of the other two sides are given.

501. If three circles whose centres are not in the same straight line intersect one another, the common chords intersect at a common point.

502. The line drawn from the vertex of either base angle of a triangle through the middle point of the median divides the opposite side into segments one of which is twice as long as the other.

503. In an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.

504. The sum of the squares of the diagonals of a trapezoid is equal to the sum of the squares of the legs, increased by twice the product of the bases.

505. To inscribe a square in a given triangle.

506. To inscribe a square in a given semicircle.

507. To inscribe in a given triangle a rectangle similar to a given rectangle.

508. To inscribe in a given semicircle a rectangle similar to a given rectangle.

509. To find the locus of a point whose distances from two given points are in a given ratio.

510. A line is drawn from a given point to a given circumference; find the locus of the point which divides the line in a given ratio.

PROBLEMS OF COMPUTATION.

1. If a post 8 feet high casts a shadow 7 feet long, how high is a steeple which at the same time casts a shadow 154 feet long?

2. The bases of two similar triangles are 6 feet and 10 feet, and the altitude of the former is 5 feet; find the altitude of the latter.

3. The sides of a triangle are 20, 15, and 12. The longest side of a similar triangle is 50; find the other two sides.

4. The perimeters of two similar polygons are 156 feet and 65 feet; if a side of the first polygon is 48 feet, find the homologous side of the second.

5. The sides of a triangle are 7, 13, and 15; is the largest angle acute, right, or obtuse?

6. The sides of a triangle are 8, 15, and 17; is the largest angle acute, right, or obtuse?

7. The sides of a triangle are 9, 16, and 18; is the largest angle acute, right, or obtuse?

8. The side of an equilateral triangle is 12; what is its altitude?

9. The altitude of an equilateral triangle is 16; find the length of a side.

10. The point E lies in the line CD and is equidistant from A and B. If the line AB cuts the line CD at an angle of 45° at a point 7 inches distant from A and 17 inches from B, find the length of EA.

11. A pole was broken 21 feet above the ground, and fell so that the top struck the ground 28 feet from the foot; find the length of the pole.

12. A ladder 26 feet long stands 10 feet from the bottom of a house, and leans against its side 3 feet below the eaves; how high is the house?

13. Find the width of a street, from a point in which a ladder 36 feet long will reach a window 28 feet high on one side, and one $25\frac{1}{2}$ feet high on the other.

14. One leg of a right triangle is 72 feet, and the difference between the hypotenuse and the other leg is 32 feet; find the lengths of the hypotenuse and the other leg.

15. Two sides of a triangle are 17 and 10; the perpendicular from their point of intersection to the third side is 8; find the length of the third side.

16. The sides of a triangle are 5, 12, and 13. Find the length of the projection of the side 5 on the side 13, and the perpendicular drawn to the side 13 from the opposite vertex.

17. The legs of a right triangle are 15 and 20 inches. Compute the lengths of the perpendicular let fall on the hypotenuse from the opposite vertex, of the segments into which the hypotenuse is divided, and of the radius of the circumscribed circle.

18. The hypotenuse of a right triangle is $37\frac{1}{2}$, and the perpendicular drawn to it from the opposite vertex is 18. Find the other two sides of the triangle, and the segments of the hypotenuse formed by the perpendicular.

19. The length of one leg of a right triangle is 12, and the length of the perpendicular drawn from the vertex of the right angle to the hypotenuse is $4\frac{8}{13}$; find the length of the hypotenuse, and also the length of the other leg.

20. In a right triangle, the projections of the legs on the hypotenuse are 9 inches and 16 inches; how long are the legs?

21. One of the legs of an isosceles right triangle is $10\sqrt{2}$; what is the length of the perpendicular dropped upon the hypotenuse from the vertex of the opposite angle.

22. The legs of an isosceles triangle are a units in length, and the base is b units; find the length of a line joining the vertex to a point of trisection of the base.

23. The base of an isosceles triangle is a, and the perpendicular drawn from an extremity of the base to the opposite side is b. Find the length of a leg.

24. If the sides of a triangle are 9, 12, and 14, find the lengths of the three medians.

25. If the sides of a triangle are 9, 12, and 14, find the length of the bisector of the largest angle.

26. If the sides of a triangle are 13, 14, and 15, find the radius of the circumscribed circle.

27. A parallelogram inscribed in a circle has two sides 20 feet long and two sides 15 feet long; find the lengths of the diagonals.

28. The diagonals of a parallelogram circumscribed about a circle are 60 inches and 80 inches in length. Find the lengths of the sides.

29. The diagonals of a trapezoid, whose lengths are 8 feet and 12 feet, divide each other into segments, which in the case of the shorter diagonal are 3 feet and 5 feet long. Find the segments of the other diagonal.

30. The bases of a circumscribed trapezoid are 8 inches and 22 inches, and the legs are equal to each other. What is the length of the diameter of the circle?

31. The bases of a trapezoid are 16 and 28, and the altitude is 9. Find the altitudes of the two triangles formed by producing the legs till they meet.

32. If each of the legs of a trapezoid is 8 feet long, and if one of the bases is 8 feet longer than the other, what is the altitude of the trapezoid?

33. The bases of an isosceles trapezoid are 9 feet and 15 feet, and each of the legs is 5 feet; find the lengths of the diagonals.

34. Two chords of a circle intersect. The segments of one are 4 feet and 6 feet, and one segment of the other is 2 feet. What is the length of the remaining segment?

35. Two chords of a circle intersect. The segments of one are 4 feet and 9 feet, and the length of the other is 15 feet. Find the segments of the second chord.

36. Two chords of a circle intersect. The segments of one are 5 feet and 8 feet, and the difference of the segments of the other is 6 feet. Find the length of the second chord.

37. One segment of a chord drawn through a point 7 inches from the centre of a circle is 4 inches. If the diameter of the circle is 15 inches, what is the length of the other segment?

38. The distance from the centre of a circle to a chord 10 inches long is 12 inches. Find the distance from the centre to a chord 24 inches long.

39. In a certain circle a chord whose length is 16 inches is at a distance of 6 inches from the centre; find the radius of a circle and the length of a chord whose distance from the centre is 4 inches.

40. In a certain circle a chord is 10 inches long while another chord twice as far from the centre is 6 inches long; find the radius of the circle and the distances of the chords from the centre.

41. If a chord AB, 5 inches long, is drawn in a circle whose diameter is 13 inches, find the perpendicular distance from B to the diameter drawn through A.

42. Through a point within a circle a chord is drawn. The segments of the chord are 6 inches and 12 inches, and the radius is 11 inches. What is the distance of the point from the centre?

43. The length of a chord of a circle is 10 feet, and its greatest distance from the subtended arc is $2\frac{5}{8}$ feet; find the radius of the circle.

44. The distance from the middle point of a chord to the middle point of its subtended arc is 12 feet, and the chord of half the arc is 18 feet; find the diameter of the circle.

45. The diameter of a circle is 24 feet, and it is divided into three equal parts; find the lengths of the chords drawn through the points of division perpendicular to the diameter.

46. Two secants intersect at a point without the circle. If the external segments are 6 and 8, while the internal segment of the former is 14, what is the internal segment of the latter?

47. From the end of a tangent 12 feet long a secant is drawn through the centre of the circle. If the external segment of the secant is 9 feet, what is the radius of the circle?

48. Through a point 15 inches from the centre of a circle two tangents are drawn. If the radius is 9 inches, find the lengths of the tangents, and also the length of the chord joining the points of contact.

49. Two tangents to a circle, each 4 feet long, meet at a point P, and include between them at P an angle of 120°; find the distance of P from the centre of the circle, and also the radius of the circle.

50. The radius of a circle is 13 inches. Through a point 5 inches from the centre a chord is drawn. What is the product of the two segments of the chord? What is the length of the shortest chord that can be drawn through the point?

51. If the radius of a circle is 5 feet, and the distance of a point P from the centre is 3 feet, what is the value of the product of the segments of any of the chords which pass through P? How long are the longest and shortest of these chords?

52. Two chords of a circle bisect each other. One of them is 10 inches long; how far is it from the centre of the circle?

53. The radii of two concentric circles are 36 inches and 39 inches. A chord of the larger circle is tangent to the smaller; how long is the chord?

54. The length of the common chord of two intersecting circles is 24, and their radii are 37 and 13; find the distance between their *centres.*

55. Two circles whose diameters are 8 inches and 12 inches are tangent externally. What is the length of their common tangent?

56. The radii of two circles are 6 inches and 13 inches, and the distance between their centres is 25 inches. Find the length of the exterior common tangent, and also of the interior common tangent.

57. On a given line 6 inches long a segment of a circle is described which contains an angle of 45°. Find the length of the radius of the circle.

58. A line *AB*, 10 inches long, is inscribed in a circle whose radius is 13 inches. From *C*, the centre of the circle, a perpendicular is let fall upon *AB* and is extended until it meets the circumference at *D*. Prove that the path traced by the middle point of the chord *AD*, as *AB* is moved from one of its inscribed positions to another, is the arc of a circle, and find the radius of the circle.

BOOK V.

AREAS OF POLYGONS. EQUIVALENT POLYGONS.

197. The *area* of a surface is the ratio of the surface to another surface, called the *unit of surface.* For example, if S is a certain surface and U is the unit of surface, the area of S is $\frac{S}{U}$. The usual unit of surface is a square, each side of which is a linear unit; as the square inch, or the square meter.

In many propositions relating to areas, the words "triangle," "rectangle," etc., are used in place of "area of triangle," "area of rectangle," etc.

Surfaces that have equal areas are said to be *equivalent.*

Two triangles are said to be *between the same parallels* when their bases are segments of the same straight line and their vertices lie in a line parallel to that line. Two parallelograms or two trapezoids are said to be *between the same parallels* when the bases of one are segments of the same parallel lines as the respective bases of the other.

Proposition 162. Theorem.

198. *Two rectangles having equal altitudes are to each other as their bases.*

CASE I. *When the bases are commensurable.*

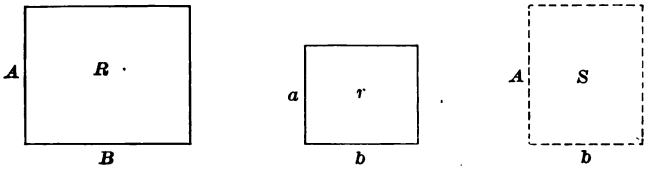
CASE II. *When the bases are incommensurable.*

Use the method of proof given in Prop. 79, and consult Prop. 55, Cor.

Cor. *Two rectangles having equal bases are to each other as their altitudes.*

Proposition 163. Theorem.

199. *Any two rectangles are to each other as the products of their bases and altitudes.*



Hypothesis. R and r are two rectangles whose respective bases are B and b, and whose respective altitudes are A and a.

Conclusion. $\frac{R}{r}=\frac{B\times A}{b\times a}$.

Proof. Construct the rect. S with a base equal to b and an altitude equal to A.

$$\text{Then } \frac{R}{S}=\frac{B}{b} \qquad (?)$$

$$\frac{S}{r}=\frac{A}{a}. \qquad (?)$$

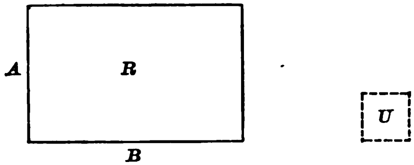
$$\therefore \frac{R}{r}=\frac{A\times B}{a\times b}. \qquad (?)$$

Q.E.D.

Scholium. It must be remembered that the product of two lines means simply the product of the numerical measures of these lines.

Proposition 164. Theorem.

200. *The area of a rectangle is equal to the product of its base and altitude.*



Hypothesis. R is a rectangle whose base and altitude are B and A respectively.

Conclusion. Area of $R = B \times A$.

Proof. Construct the square U with each side equal to the linear unit. Then U is the unit of surface. (?)

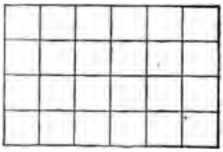
$$\frac{R}{U} = \frac{B \times A}{1 \times 1} = B \times A. \qquad (?)$$

$$\therefore \text{ Area of } R = B \times A. \qquad (?)$$

Q.E.D.

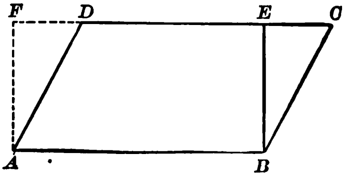
COR. *The area of a square is equal to the square of its side.*

SCHOLIUM. When the base and altitude are of such lengths that the linear unit is contained an exact number of times in each, this proposition is rendered evident by dividing the rectangle into squares, each equal to the unit of surface. For example, if the base contains 6 linear units, and the altitude 4, the rectangle may be divided into 24 squares, each equal to the unit of surface; that is, the area is equal to 6 × 4 units of surface.



Proposition 165. Theorem.

201. *The area of a parallelogram is equal to the product of its base and altitude.*



Hypothesis. $ABCD$ is a ▱ whose base and altitude are AB and BE respectively.

Conclusion. Area of ▱ $ABCD = AB \times BE$.

Proof. Draw $AF \parallel$ to BE to meet CD produced at F.

Then $ABEF$ is a rectangle with the same base and altitude as the ▱ $ABCD$. (?)

$\triangle AFD = \triangle BEC$. (?)

Then ▱ $ABCD \Leftrightarrow$ rect. $ABEF$. (?)

But the area of rect. $ABEF = AB \times BE$. (?)

$\therefore$ area of ▱ $ABCD = AB \times BE$. (?)

Q.E.D.

Cor. I. *Parallelograms having equal bases and equal altitudes are equivalent.*

Cor. II. *Parallelograms upon equal bases and between the same parallels are equivalent.*

Cor. III. *Any two parallelograms are to each other as the products of their bases and altitudes.*

Cor. IV. *Two parallelograms having equal altitudes are to each other as their bases.*

Cor. V. *Two parallelograms having equal bases are to each other as their altitudes.*

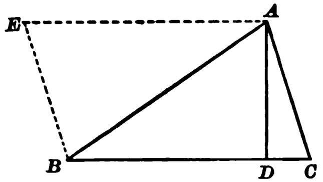
Ex. 511. Upon the same base, to construct a rectangle equivalent to a given parallelogram.

Ex. 512. Upon the same base, to construct a rhombus equivalent to a given parallelogram.

Ex. 513. To construct a parallelogram equivalent to a given parallelogram, and having an angle equal to a given angle.

Proposition 166. Theorem.

202. *The area of a triangle is equal to one half the product of its base and altitude.*



Hint. Construct the ▱ $BCAE$ by drawing $AE \parallel$ to CB and $BE \parallel$ to CA.

Cor. I. *A triangle is equivalent to one half of a parallelogram having the same base and altitude.*

Cor. II. *If a triangle and a parallelogram are upon equal bases and between the same parallels, the triangle is equivalent to one half of the parallelogram.*

Cor. III. *Triangles having equal bases and equal altitudes are equivalent.*

Cor. IV. *Triangles upon equal bases and between the same parallels are equivalent.*

Cor. V. *Any two triangles are to each other as the products of their bases and altitudes.*

Cor. VI. *Two triangles having equal altitudes are to each other as their bases.*

Cor. VII. *Two triangles having equal bases are to each other as their altitudes.*

Ex. 514. A median of a triangle divides it into two equivalent parts.

Ex. 515. E is any point in the median AD of the triangle ABC; prove that the triangles ABE and ACE are equivalent.

Ex. 516. The diagonals of a parallelogram divide it into four equivalent triangles.

Ex. 517. In the parallelogram $ABCD$, E and F are the middle points of AB and CD respectively, and G is any point in EF; prove that the triangles GBC and GAD are equivalent.

Ex. 518. To cut off one fourth of the area of a given triangle by a line drawn through either vertex.

Ex. 519. Upon the same base, to construct a right triangle equivalent to a given triangle.

Ex. 520. To construct a parallelogram equivalent to a given triangle, and having an angle equal to a given angle.

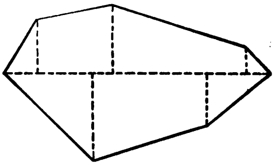
Proposition 167. Theorem.

203. *The area of a trapezoid is equal to the product of its altitude and one half the sum of its bases.*

Hint. Draw a diagonal.

Cor. *The area of a trapezoid is equal to the product of its altitude and median.*

Scholium. The area of any polygon may be found by drawing diagonals from one of its vertices, and then finding the sum of the areas of the triangles thus formed.



A more usual method, however, consists in drawing the longest diagonal of the polygon, and then drawing perpendiculars to it from the remaining vertices. The polygon is thus divided into right triangles and trapezoids, and the sum of the areas of these figures is the area of the polygon.

Ex. 521. E is the middle point of BC, one of the legs of the trapezoid $ABCD$, and through E a line is drawn parallel to AD, meeting AB and CD at F and G respectively; prove that the parallelogram $AFGD$ is equivalent to the trapezoid.

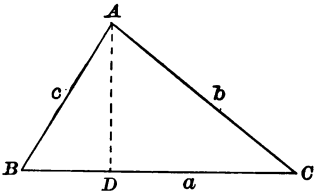
Ex. 522. To construct a parallelogram equivalent to a given trapezoid.

Ex. 523. The sides AD and BC of the trapezoid $ABCD$ when produced meet at E; prove that the triangles EAC and EBD are equivalent.

Ex. 524. The area of a trapezoid is equal to the product of one of the legs and the distance of this leg from the middle point of the other leg.

Proposition 168. Problem.

204. *To compute the area of a triangle in terms of its sides.*



Let a, b, and c denote respectively the sides BC, AC, and AB of the $\triangle ABC$, and let S denote the area of the triangle.

To compute S in terms of a, b, and c.

Draw the altitude AD.

Either B or C is an acute $\angle$. (?)

Let B be an acute $\angle$.

$$\text{Then } b^2 = a^2 + c^2 - 2a \times BD. \qquad (?)$$

$$\therefore BD = \frac{a^2 + c^2 - b^2}{2a}.$$

$$\overline{AD}^2 = c^2 - \overline{BD}^2 = c^2 - \frac{(a^2 + c^2 - b^2)^2}{4a^2} \qquad (?)$$

$$= \frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4a^2}$$

$$= \frac{(2ac + a^2 + c^2 - b^2)(2ac - a^2 - c^2 + b^2)}{4a^2}$$

$$= \frac{[(a + c)^2 - b^2][b^2 - (a - c)^2]}{4a^2}$$

$$= \frac{(a + c + b)(a + c - b)(b + a - c)(b - a + c)}{4a^2}$$

Let $a + b + c = 2s$.

Then $a + c - b = 2(s - b)$, $b + a - c = 2(s - c)$, and $b - a + c = 2(s - a)$. (?)

$$\therefore \overline{AD}^2 = \frac{16s(s - a)(s - b)(s - c)}{4a^2}. \qquad (?)$$

$$\therefore AD = \frac{2\sqrt{s(s - a)(s - b)(s - c)}}{a}. \qquad (?)$$

Now $S = \frac{1}{2}a \times AD$. (?)

$$\therefore S = \sqrt{s(s - a)(s - b)(s - c)}. \qquad (?)$$

Q.E.F.

Ex. 525. To compute the area of an equilateral triangle in terms of its side.

Ex. 526. To compute a median of a triangle in terms of its sides.

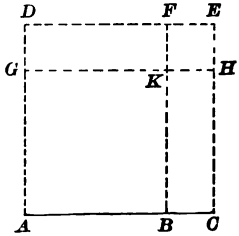
Ex. 527. To compute the bisector of an angle of a triangle in terms of its sides.

Ex. 528. To compute the radius of the circle circumscribed about a triangle in terms of the sides of the triangle.

Ex. 529. The area of a triangle is equal to the product of the three sides divided by four times the radius of the circumscribed circle.

Proposition 169. Theorem.

205. *The square constructed on the sum of two lines is equivalent to the sum of the squares on the two lines, increased by twice the rectangle contained by the lines.*



Hypothesis. The line AC is the sum of the lines AB and BC.

Conclusion. The square on AC is equivalent to the sum of the squares on AB and BC, increased by twice the rectangle contained by AB and BC.

Proof. On AC construct the square $ACED$.

Through B draw $BF \parallel$ to AD.

On AD lay off $AG = AB$, and draw $GH \parallel$ to AC.

The square $ABED$ is thus divided into four parts, $ABKG$, $KHEF$, $BCHK$, and $GKFD$, all of which are rectangles. (?)

$ABKG$ is a square on AC. (?)

$KH = BC$. (?)

$HE = BC$. (?)

$\therefore$ $KHEF$ is equal to a square on BC. (?)

$CH = AB$. (?)

Then $BCHK$ is a rectangle whose dimensions are AB and BC.

$GK = AB$. (?)

$GD = BC$. (?)

Then *GKFD* is a rectangle whose dimensions are *AB* and *BC*.

∴ the square *ACED* is equivalent to the sum of the squares on *AB* and *BC*, increased by twice the rectangle contained by *AB* and *BC*.

Q.E.D.

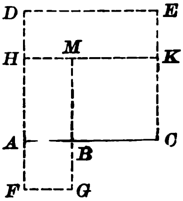
Scholium. This theorem corresponds to the algebraic formula

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Proposition 170. Theorem.

206. *The square constructed on the difference of two lines is equivalent to the sum of the squares constructed on the two lines, diminished by twice the rectangle contained by the lines.*

Hint. The line *BC* is the difference of the lines *AC* and *AB*.



Scholium. This theorem corresponds to the algebraic formula

$$(a-b)^2 = a^2 - 2ab + b^2.$$

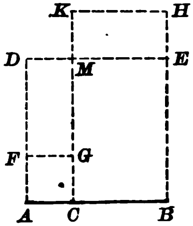
Ex. 530. The lines *AB* and *CD* intersect at *E*, and *AD* and *BC* are parallel; prove that the triangles *AEC* and *BED* are equivalent.

Ex. 531. In any right triangle, the product of the legs is equal to the product of the hypotenuse and the perpendicular drawn to the hypotenuse from the vertex of the right angle.

Ex. 532. The line joining the middle points of the bases of a trapezoid divides it into two equivalent parts.

Proposition 171. Theorem.

207. *The difference of the squares constructed on two lines is equivalent to the rectangle contained by the sum and the difference of the lines.*



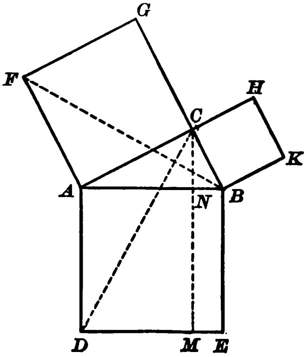
HINT. On AB construct the square $ABED$, and on AC construct the square $ACGF$. Produce BE to H, making $EH = AC$. Draw $HK \parallel$ to BA to meet CG produced at K.

SCHOLIUM. This theorem corresponds to the algebraic formula

$$a^2 - b^2 = (a + b)(a - b).$$

Proposition 172. Theorem.

208. *The square constructed on the hypotenuse of a right triangle is equivalent to the sum of the squares on the two legs.*



Hypothesis. AB is the hypotenuse of the rt. $\triangle ABC$, and $ABED$, $ACGF$, and $CBKH$ are the squares constructed on AB, AC, and BC respectively.

Conclusion. Square $ABED$ ≎ square $ACGF$ + square $CBKH$.

Proof. Through C draw $CM \parallel$ to AD, intersecting AB at N; also draw CD and FB.

AH and BG are straight lines. (?)
$\angle CAD = \angle FAB$. (?)
Then $\triangle CAD = \triangle FAB$. (?)
Rect. $ADMN$ ≎ $2 \triangle CAD$. (?)
Square $ACGF$ ≎ $2 \triangle FAB$. (?)
∴ rect. $ADMN$ ≎ square $ACGF$. (?)

In like manner, by drawing CE and AK, it may be proved that rect. $NMEB$ ≎ square $CBKH$.

∴ square $ADEB$ ≎ square $ACGF$ + square $CBKH$. (?)

Q.E.D.

Cor. I. *The square constructed on either leg of a right triangle is equivalent to the difference of the squares on the hypotenuse and the other leg.*

Cor. II. *The square constructed on the diagonal of a square is double the square itself.*

Scholium. This proposition is said to have been discovered by Pythagoras, and it is commonly known as the Pythagorean Proposition. The preceding demonstration is the celebrated forty-seventh proposition of Euclid.

Since the area of a square is equal to the square of its side, the areas of the squares may be expressed by $\overline{AB}^2$, $\overline{AC}^2$, and $\overline{BC}^2$. Then $\overline{AB}^2$ ≎ $\overline{AC}^2 + \overline{BC}^2$.

Ex. 533. If the three altitudes of a triangle are equal, the triangle is equilateral.

Ex. 534. Two equivalent triangles on the same base and on the same side of it are between the same parallels.

Ex. 535. In the figure of Proposition 172,

(i.) Prove that the points F, C, and K lie in the same straight line.

(ii.) Prove that the straight lines AG and BH are parallel.

(iii.) Prove that the straight lines BF and CD are perpendicular to each other.

(iv.) Prove that the sum of the perpendiculars drawn from F and K to AB produced is equal to AB.

(v.) Prove that FG, MC, and KH when produced intersect at a common point.

(vi.) Prove that the triangles ABC, GCH, DAF, and EBK are equivalent.

(vii.) Prove that $\overline{CD}^2 - \overline{CE}^2 = \overline{AC}^2 - \overline{BC}^2$.

(viii.) Prove that $\overline{DF}^2 + \overline{EK}^2 = 5\,\overline{AB}^2$.

Ex. 536. State and prove the converse of Prop. 172.

Proposition 173. Problem.

209. *To construct a square equivalent to the sum of two given squares.*

Consult Prop. 172.

Proposition 174. Problem.

210. *To construct a square equivalent to the sum of three or more given squares.*

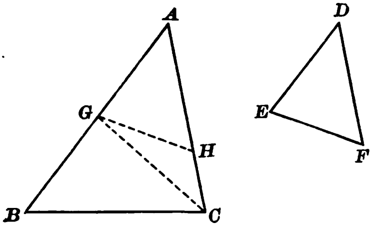
Construct a square equivalent to the sum of two of the given squares. Then construct a square equivalent to the sum of this square and another one of the given squares, and so on.

Proposition 175. Problem.

211. *To construct a square equivalent to the difference of two given squares.*

Proposition 176. Theorem.

212. *Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.*



Hypothesis. In the $\triangle s\ ABC$ and DEF, $\angle A = \angle D$.

Conclusion. $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \times AC}{DE \times DF}$.

Proof. On AB lay off $AG = DE$, and on AC lay off $AH = DF$. Draw GH and GC.

$$\triangle AGH = \triangle DEF. \quad (?)$$

$$\frac{\triangle ABC}{\triangle AGC} = \frac{AB}{AG}. \quad (?)$$

$$\frac{\triangle AGC}{\triangle AGH} = \frac{AC}{AH}. \quad (?)$$

$$\therefore \frac{\triangle ABC}{\triangle AGH} = \frac{AB \times AC}{AG \times AH}. \quad (?)$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{AB \times AC}{DE \times DF}. \quad (?)$$

Q.E.D.

Ex. 537. If two equivalent triangles have an angle of one equal to an angle of the other, the sides about the equal angles are reciprocally proportional.

Ex. 538. If two triangles have two sides of one equal respectively to two sides of the other and the included angles supplementary, the triangles are equivalent.

Ex. 539. Two triangles which have an angle of one equal to the supplement of an angle of the other are to each other as the products of the sides including the supplementary angles.

Ex. 540. If two triangles have an angle of one equal to an angle of the other, and a second angle of one supplementary to a second angle of the other, the sides about the third angles are proportional.

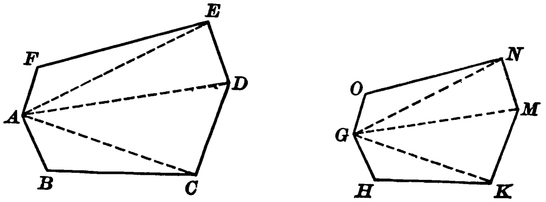
Proposition 177. Theorem.

213. *Similar triangles are to each other as the squares of their homologous sides.*

Consult Prop. 166, Cor. V. and Prop. 140.

Proposition 178. Theorem.

214. *Similar polygons are to each other as the squares of their homologous sides.*



Hypothesis. AB and GH are homologous sides of the similar polygons $ABCDEF$ and $GHKMNO$. Let their areas be denoted by S and s.

Conclusion. $\dfrac{S}{s}=\dfrac{\overline{AB}^2}{\overline{GH}^2}$.

Proof. From the homologous vertices A and G draw the diagonals AC, AD, AE, GK, GM, and GN.

$\triangle$s ABC and ACD are respectively similar to $\triangle$s GHK and GKM. (?)

$$\therefore \frac{\triangle ABC}{\triangle GHK} = \frac{\overline{AC}^2}{\overline{GK}^2}, \text{ and } \frac{\triangle ACD}{\triangle GKM} = \frac{\overline{AC}^2}{\overline{GK}^2}. \qquad (?)$$

$$\therefore \frac{\triangle ABC}{\triangle GHK} = \frac{\triangle ACD}{\triangle GKM}. \qquad (?)$$

In like manner it may be proved that

$$\frac{\triangle ACD}{\triangle GKM} = \frac{\triangle ADE}{\triangle GMN} = \frac{\triangle AEF}{\triangle GNO}.$$

Then

$$\frac{\triangle ABC + \triangle ACD + \triangle ADE + \triangle AEF}{\triangle GHK + \triangle GKM + \triangle GMN + \triangle GNO} = \frac{\triangle ABC}{\triangle GHK}. \qquad (?)$$

$$\text{But } \frac{\triangle ABC}{\triangle GHK} = \frac{\overline{AB}^2}{\overline{GH}^2}. \qquad (?)$$

$$\therefore \frac{S}{s} = \frac{\overline{AB}^2}{\overline{GH}^2}. \qquad (?)$$

Q.E.D.

Ex. 541. Similar polygons are to each other as the squares of their perimeters.

Ex. 542. In the triangle ABC, the medians AD and BE intersect at F; prove that the triangle DEF is equivalent to one fourth of the triangle ABF.

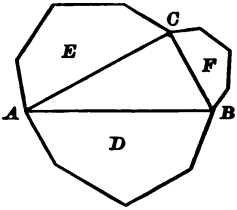
Ex. 543. AD and BC are the parallel sides of the trapezoid $ABCD$, whose diagonals intersect at E; prove that the triangles AED and BEC are to each other as the squares of AE and EC.

Ex. 544. Similar trapezoids are to each other as the products of their legs.

Ex. 545. To construct an isosceles triangle equivalent to a given triangle, and having its vertical angle equal to the vertical angle of the given triangle.

Proposition 179. Theorem.

215. *If similar polygons are constructed on the three sides of a right triangle as homologous sides, the polygon constructed on the hypotenuse is equivalent to the sum of the polygons constructed on the legs.*



Hypothesis. AB is the hypotenuse of the rt. $\triangle ABC$, and D, E, and F are similar polygons constructed on AB, AC, and BC respectively as homologous sides.

Conclusion. $D \Leftrightarrow E + F$.

Proof.

$$\frac{E}{D} = \frac{\overline{AC}^2}{\overline{AB}^2}. \qquad (?)$$

$$\frac{F}{D} = \frac{\overline{BC}^2}{\overline{AB}^2}. \qquad (?)$$

$$\text{Then } \frac{E+F}{D} = \frac{\overline{AC}^2 + \overline{BC}^2}{\overline{AB}^2}. \qquad (?)$$

[To be completed by the student.]

COR. *If similar polygons are constructed on the three sides of a right triangle as homologous sides, the polygon constructed on either leg is equivalent to the difference of the polygons constructed on the hypotenuse and the other leg.*

Ex. 546. In the quadrilateral $ABCD$, the diagonals intersect at E, and the triangles AED and BEC are equivalent; prove that AB is parallel to DC.

Proposition 180. Problem.

216. *To construct a polygon similar to two given similar polygons and equivalent to their sum.*

Consult Prop. 179 and Prop. 143.

Proposition 181. Problem.

217. *To construct a polygon similar to two given similar polygons and equivalent to their difference.*

Ex. 547. If two isosceles triangles are in the same ratio as the squares of their bases, the triangles are similar.

Ex. 548. Similar triangles are to each other as the squares of the radii of their circumscribed circles.

Ex. 549. Similar triangles are to each other as the squares of the radii of their inscribed circles.

Ex. 550. From the extremities of the hypotenuse AB of the right triangle ABC, the lines AD and BE are drawn perpendicular to AB, meeting BC and AC produced at D and E respectively; prove that the triangles ABC and CDE are equivalent.

Ex. 551. If E and F are any two points in the sides AB and AD respectively of the parallelogram $ABCD$, prove that the triangles BCF and CDE are equivalent.

Ex. 552. If two straight lines are drawn from the middle point of one of the legs of a trapezoid to the extremities of the other leg, the triangle thus formed is equivalent to one half of the trapezoid.

Ex. 553. If two equivalent triangles on the same base are cut by a line parallel to the base, the triangles thus formed are equivalent.

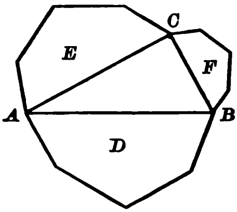
Ex. 554. If a straight line is drawn joining the middle points of two adjacent sides of a parallelogram, the triangle thus formed is equivalent to one eighth of the parallelogram.

Ex. 555. If E and F are the middle points of the sides BC and CD respectively of the parallelogram $ABCD$, prove that the triangle AEF is equivalent to three eighths of the parallelogram.

Ex. 556. If from the middle point of one of the legs of a right triangle a perpendicular is drawn to the hypotenuse, the difference of the squares on the segments of the hypotenuse is equivalent to the square on the other leg.

Proposition 179. Theorem.

215. *If similar polygons are constructed on the three sides of a right triangle as homologous sides, the polygon constructed on the hypotenuse is equivalent to the sum of the polygons constructed on the legs.*



Hypothesis. AB is the hypotenuse of the rt. $\triangle ABC$, and D, E, and F are similar polygons constructed on AB, AC, and BC respectively as homologous sides.

Conclusion. $D \Leftrightarrow E + F$.

Proof.

$$\frac{E}{D} = \frac{\overline{AC}^2}{\overline{AB}^2}. \qquad (?)$$

$$\frac{F}{D} = \frac{\overline{BC}^2}{\overline{AB}^2}. \qquad (?)$$

$$\text{Then } \frac{E+F}{D} = \frac{\overline{AC}^2 + \overline{BC}^2}{\overline{AB}^2}. \qquad (?)$$

[To be completed by the student.]

Cor. *If similar polygons are constructed on the three sides of a right triangle as homologous sides, the polygon constructed on either leg is equivalent to the difference of the polygons constructed on the hypotenuse and the other leg.*

Ex. 546. In the quadrilateral $ABCD$, the diagonals intersect at E, and the triangles AED and BEC are equivalent; prove that AB is parallel to DC.

Proposition 180. Problem.

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Ex. 549. Similar triangles are to each other as the squares of the radii of their inscribed circles.

Ex. 550. From the extremities of the hypotenuse AB of the right triangle ABC, the lines AD and BE are drawn perpendicular to AB, meeting BC and AC produced at D and E respectively; prove that the triangles ABC and CDE are equivalent.

Ex. 551. If E and F are any two points in the sides AB and AD respectively of the parallelogram $ABCD$, prove that the triangles BCF and CDE are equivalent.

Ex. 552. If two straight lines are drawn from the middle point of one of the legs of a trapezoid to the extremities of the other leg, the triangle thus formed is equivalent to one half of the trapezoid.

Ex. 553. If two equivalent triangles on the same base are cut by a line parallel to the base, the triangles thus formed are equivalent.

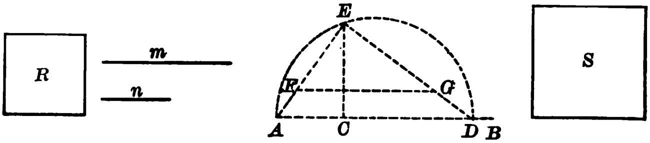
Ex. 554. If a straight line is drawn joining the middle points of two adjacent sides of a parallelogram, the triangle thus formed is equivalent to one eighth of the parallelogram.

Ex. 555. If E and F are the middle points of the sides BC and CD respectively of the parallelogram $ABCD$, prove that the triangle AEF is equivalent to three eighths of the parallelogram.

Ex. 556. If from the middle point of one of the legs of a right triangle a perpendicular is drawn to the hypotenuse, the difference of the squares on the segments of the hypotenuse is equivalent to the square on the other leg.

Proposition 182. Problem.

218. *To construct a square whose ratio to a given square shall equal the ratio of two given lines.*



To construct a square whose ratio to the square R shall equal the ratio $\frac{m}{n}$.

Construction. On any indefinite line AB lay off $AC = n$ and $CD = m$.

On AD as a diameter describe a semi-circumference.

At C erect $CE \perp$ to AD, meeting the semi-circumference at E, and draw EA and ED.

On EA (produced if necessary) lay off EF equal to a side of R.

Draw $FG \parallel$ to AD, meeting ED at G.

Construct the square S with a side equal to EG.

Then S is the required square.

Proof.

$$\frac{EG}{EF} = \frac{ED}{EA}. \qquad (?)$$

$$\therefore \frac{\overline{EG}^2}{\overline{EF}^2} = \frac{\overline{ED}^2}{\overline{EA}^2}. \qquad (?)$$

$$\text{But } \frac{\overline{ED}^2}{\overline{EA}^2} = \frac{CD}{AC} = \frac{m}{n}. \qquad (?)$$

$$\therefore \frac{\overline{EG}^2}{\overline{EF}^2} = \frac{m}{n}. \qquad (?)$$

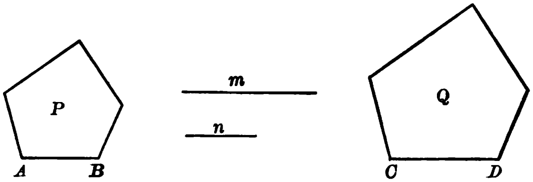
$$\therefore \frac{S}{R} = \frac{m}{n}. \qquad (?)$$

Q.E.F.

Ex. 557. To draw two straight lines having the same ratio as the areas of two given squares.

Proposition 183. Problem.

219. *To construct a polygon similar to a given polygon, and having a ratio to it equal to the ratio of two given lines.*



To construct a polygon similar to P, and having a ratio to P equal to the ratio $\frac{m}{n}$.

Construction. On AB, a side of P, construct a square.

Then find CD, the side of a square whose ratio to the square constructed on AB equals $\frac{m}{n}$.

Upon CD, homologous to AB, construct the polygon Q similar to P.

Then Q is the required polygon.

[The proof is left to the student.]

Ex. 558. Upon the same base, to construct an isosceles triangle equivalent to a given triangle.

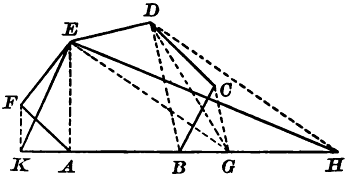
Ex. 559. Upon the same base, to construct an isosceles triangle equivalent to a given square.

Ex. 560. Upon the same base, to construct a rectangle equivalent to a given triangle.

Ex. 561. To construct an isosceles right triangle equivalent to a given square.

Proposition 184. Problem.

220. *To construct a triangle equivalent to a given polygon.*



To construct a △ equivalent to the polygon *ABCDEF*.

Construction. From any vertex *B* draw the diagonal *BD* cutting off the △ *BCD*.

Draw *CG* ∥ to *DB* to meet *AB* produced at *G*.

Draw *DG*.

The polygon *AGDEF* has one side less than the polygon *ABCDEF*.

By repeating this process a series of polygons is obtained, each one of which has one side less than the preceding one.

The △ *EKH*, the last of this series, is the required △.

Proof. △ *GDB* ≎ △ *CDB*. (?)

∴ polygon *AGDEF* ≎ polygon *ABCDEF*. (?)

In like manner it may be proved that each polygon of the series is equivalent to the preceding one.

∴ △ *EKH* ≎ polygon *ABCDEF*.

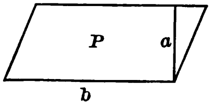
Q.E.F.

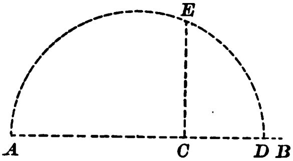
Ex. 562. To construct a triangle equivalent to a given triangle, and having one side equal to a given line and an adjacent angle equal to a given angle of the given triangle.

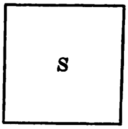
Ex. 563. To construct a parallelogram equivalent to a given triangle, and having an angle equal to a given angle.

Proposition 185. Problem.

221. *To construct a square equivalent to a given parallelogram.*







To construct a square equivalent to the ▱ P, whose base and altitude are b and a respectively.

Construction. On any indefinite line AB lay off $AC = b$ and $CD = a$.

On AD as a diameter describe a semi-circumference.

At C erect $CE \perp$ to AD, meeting the semi-circumference at E.

Construct the square S with a side equal to CE.

Then S is the required square.

[The proof is left to the student.]

Proposition 186. Problem.

222. *To construct a square equivalent to a given triangle.*

The construction is similar to that of Prop. 185. AC and CD should be taken equal either to the base and one half the altitude of the triangle or to the altitude and one half the base.

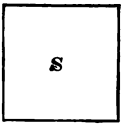
Proposition 187. Problem.

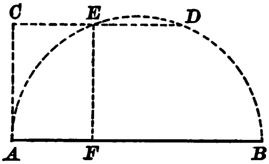
223. *To construct a square equivalent to a given polygon.*

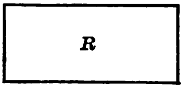
Consult Prop. 184 and Prop. 186.

Proposition 188. Problem.

224. *To construct a rectangle equivalent to a given square, and having the sum of its base and altitude equal to a given line.*







To construct a rectangle equivalent to the square *S*, and having the sum of its base and altitude equal to the line *AB*.

Construction. On *AB* as a diameter describe a semicircumference.

At *A* erect *AC* ⊥ to *AB*, equal to a side of *S*.

Through *C* draw *CD* ‖ to *AB*, intersecting the semicircumference at *E*.

Draw *EF* ⊥ to *AB*.

Construct the rect. *R* with a base equal to *FB* and an altitude equal to *AF*.

Then *R* is the required rectangle.

[The proof is left to the student.]

Query. Under what condition is the problem impossible?

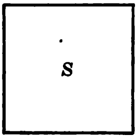
Scholium. The following problem has the same construction:

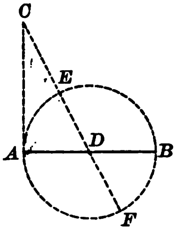
To construct two lines when their sum and product are given.

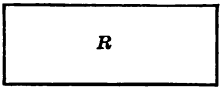
Ex. 564. To construct a parallelogram equivalent to a given square, and having the sum of its base and altitude equal to a given line and an angle equal to a given angle.

Proposition 189. Problem.

225. *To construct a rectangle equivalent to a given square, and having the difference of its base and altitude equal to a given line.*







To construct a rectangle equivalent to the square *S*, and having the difference of its base and altitude equal to the line *AB*.

Construction. On *AB* as a diameter describe a circumference.

At *A* erect *AC* ⊥ to *AB*, equal to a side of *S*.

Through *C* and *D*, the centre of the circle, draw the secant *CF* intersecting the circumference at *E* and *F*.

Construct the rect. *R* with a base equal to *CF* and an altitude equal to *CE*.

Then *R* is the required rectangle.

[The proof is left to the student.]

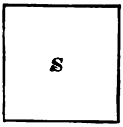
Scholium. The following problem has the same construction:

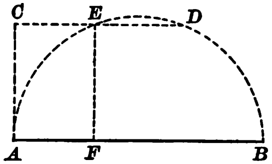
To construct two lines when their difference and product are given.

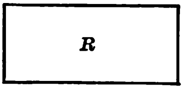
Ex. 565. To construct a parallelogram equivalent to a given square, and having the difference of its base and altitude equal to a given line and an angle equal to a given angle.

Proposition 188. Problem.

224. *To construct a rectangle equivalent to a given square, and having the sum of its base and altitude equal to a given line.*







To construct a rectangle equivalent to the square S, and having the sum of its base and altitude equal to the line AB.

Construction. On AB as a diameter describe a semi-circumference.

At A erect $AC \perp$ to AB, equal to a side of S.

Through C draw $CD \parallel$ to AB, intersecting the semi-circumference at E.

Draw $EF \perp$ to AB.

Construct the rect. R with a base equal to FB and an altitude equal to AF.

Then R is the required rectangle.

[The proof is left to the student.]

QUERY. Under what condition is the problem impossible?

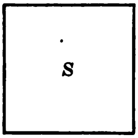
SCHOLIUM. The following problem has the same construction:

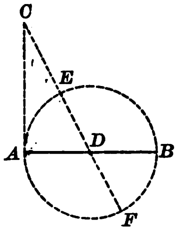
To construct two lines when their sum and product are given.

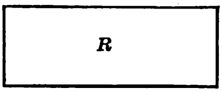
Ex. 564. To construct a parallelogram equivalent to a given square, and having the sum of its base and altitude equal to a given line and an angle equal to a given angle.

Proposition 189. Problem.

225. *To construct a rectangle equivalent to a given square, and having the difference of its base and altitude equal to a given line.*







To construct a rectangle equivalent to the square S, and having the difference of its base and altitude equal to the line AB.

Construction. On AB as a diameter describe a circumference.

At A erect $AC \perp$ to AB, equal to a side of S.

Through C and D, the centre of the circle, draw the secant CF intersecting the circumference at E and F.

Construct the rect. R with a base equal to CF and an altitude equal to CE.

Then R is the required rectangle.

[The proof is left to the student.]

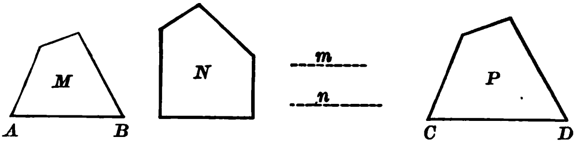
SCHOLIUM. The following problem has the same construction:

To construct two lines when their difference and product are given.

Ex. 565. To construct a parallelogram equivalent to a given square, and having the difference of its base and altitude equal to a given line and an angle equal to a given angle.

Proposition 190. Problem.

226. *To construct a polygon similar to a given polygon and equivalent to another given polygon.*



To construct a polygon similar to the polygon M and equivalent to the polygon N.

Construction. Construct m and n, the sides of squares equivalent to M and N respectively.

Find CD, a fourth proportional to m, n, and AB, a side of M.

Upon CD, homologous to AB, construct the polygon P similar to M.

Then P is the required polygon.

Proof.

$$\frac{M}{P}=\frac{\overline{AB}^2}{\overline{CD}^2}. \qquad (?)$$

$$\frac{AB}{CD}=\frac{m}{n}. \qquad (?)$$

$$\therefore \frac{\overline{AB}^2}{\overline{CD}^2}=\frac{m^2}{n^2}. \qquad (?)$$

$$\therefore \frac{M}{P}=\frac{m^2}{n^2}. \qquad (?)$$

$$\text{But } M \Bumpeq m^2 \text{ and } N \Bumpeq n^2. \qquad (?)$$

$$\therefore \frac{M}{P}=\frac{M}{N}. \qquad (?)$$

$$\therefore P \Bumpeq N. \qquad (?)$$

Q.E.F.

EXERCISES.

566. The area of a rhombus is equal to one half the product of its diagonals.

567. The area of a polygon circumscribed about a circle is equal to one half the product of its perimeter and the radius of the circumscribed circle.

568. If from any point within a parallelogram lines are drawn to the four vertices, the sum of either pair of triangles having parallel bases is equivalent to one half the parallelogram.

569. If from any point of either diagonal of a parallelogram lines are drawn to the opposite vertices, the parallelogram is divided into two pairs of equivalent triangles.

570. If from the middle point of either diagonal of a quadrilateral lines are drawn to the opposite vertices, the quadrilateral is divided into two equivalent parts.

571. $ABCD$ is a parallelogram, and E is any point in the diagonal AC produced; prove that the triangles EBC and EDC are equivalent.

572. In the right triangle ABC, AB is the hypotenuse, and the angle A is 30°; prove that the area of ABC is equal to $\frac{1}{4} AB \times AC$.

573. To construct an isosceles triangle equivalent to a given triangle, and having its base equal to a given line.

574. Upon a given straight line as a base, to construct a rectangle equivalent to a given rectangle.

575. Upon a given straight line as a base, to construct a parallelogram equivalent to a given parallelogram, and having its angles equal respectively to the angles of the given parallelogram.

576. To construct a parallelogram equivalent to a given parallelogram, and having two adjacent sides equal to two given lines. [When is the problem impossible?]

577. To construct a rhombus equivalent to a given parallelogram, and having each of its sides equal to a given line. [When is the problem impossible?]

578. Every straight line drawn through the middle point of a diagonal of a parallelogram divides it into two equal parts.

579. If through the middle point of the median of a trapezoid a line is drawn cutting the bases, the trapezoid is divided into two equivalent parts.

580. To divide a given parallelogram into two equivalent parts by drawing a straight line through a given point within the parallelogram.

581. To divide a given parallelogram into two equivalent parts by drawing a straight line through a given point in one of the sides.

582. To divide a given parallelogram into two equivalent parts by drawing a straight line parallel to one of the sides.

583. To divide a given parallelogram into two equivalent parts by drawing a straight line parallel to a given straight line.

584. To divide a given parallelogram into two equivalent parts by drawing a straight line perpendicular to one of the sides.

585. To divide a given trapezoid into two equivalent parts by drawing a straight line through a given point in one of the bases.

586. To divide a given trapezoid into two equivalent parts by drawing a straight line perpendicular to the bases.

587. If one diagonal of a quadrilateral bisects the other, it divides the quadrilateral into two equivalent triangles.

588. Two equivalent triangles have a common base, and lie on opposite sides of it; prove that the straight line joining their vertices is bisected by the base (produced if necessary).

589. The diagonals of a quadrilateral are perpendicular to each other; prove that the rectangle contained by the diagonals is double the quadrilateral.

590. Any point D in the base BC of a triangle ABC is joined with the vertex A by a straight line; from E, the middle point of the base, a line is drawn parallel to DA to meet one of the sides of the triangle at F. Prove that the straight line DF divides the triangle into two equivalent parts.

591. To divide a given triangle into two equivalent parts by drawing a straight line through a given point in one of the sides.

592. The figure formed by joining consecutively the middle points of the sides of a quadrilateral is equivalent to one half the quadrilateral.

593. If two parallelograms have two sides of one equal respectively to two sides of the other, and the included angles supplementary, the parallelograms are equivalent.

594. To construct a triangle equivalent to a given triangle, and having its base in the same straight line and its vertex at a given point.

595. To construct a triangle equivalent to a given triangle, and having its base equal to a given line.

596. To construct a triangle equivalent to a given triangle, and having two sides equal to two given lines.

597. To construct a triangle equivalent to a given triangle, and having its altitude equal to a given line.

598. To construct an equilateral triangle equivalent to a given triangle.

599. To construct an isosceles triangle equivalent to a given triangle, and having a leg equal to a given line.

600. To construct a right triangle equivalent to a given triangle, and having a leg equal to a given line.

601. To construct a right triangle equivalent to a given triangle, and having its hypotenuse equal to a given line.

602. To construct a triangle equivalent to a given square, and having its base equal to a given line and an angle adjacent to the base equal to a given angle.

603. To construct an equilateral triangle equivalent to a given square.

604. To construct a rectangle equivalent to a given square, and having its base equal to a given line.

605. To construct a rhombus equivalent to a given parallelogram, and having a diagonal coinciding with a diagonal of the parallelogram.

606. If D is a point in the base BC of the isosceles triangle ABC, the difference of the squares on AB and AD is equivalent to the rectangle contained by BD and CD.

607. If a straight line is bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced together with the square on half the line bisected is equivalent to the square on the straight line which is made up of the half and the part produced.

608. If a straight line is bisected and produced to any point, the sum of the squares on the whole line thus produced and on the part produced is equivalent to twice the sum of the squares on half the line bisected and on the line made up of the half and the part produced.

609. If in the triangle ABC the median drawn from the vertex C is equal to one half the side AB, prove that the area of the triangle is equal to one half the product of AC and BC.

610. If straight lines are drawn from the point of intersection of the medians of a triangle to the vertices, the three triangles thus formed are equivalent.

611. To find a point within a triangle such that the straight lines drawn from it to the three vertices will divide the triangle into three equivalent parts.

612. *D* and *E* are the middle points of the sides *AB* and *AC* respectively of the triangle *ABC*, and *BE* and *CD* intersect at *F*; prove that the triangle *BFC* is equivalent to the quadrilateral *ADFE*.

613. *E* is a point within the parallelogram *ABCD*; prove that the difference of the triangles *ABE* and *ADE* is equivalent to the triangle *ACE*.

614. *E* is a point in the diagonal *AC* of the parallelogram *ABCD*; through *E* the lines *FG* and *HK* are drawn parallel to *AB* and *AD* respectively. Prove that the parallelograms *EFDK* and *EHBG* are equivalent.

615. If a triangle is constructed having two sides and the included angle equal respectively to the diagonals of a quadrilateral and either angle included between them, the triangle and the quadrilateral are equivalent.

616. *ABCD* is a parallelogram, and *E* and *F* are the middle points of *AD* and *BC* respectively; if G is any point in *EF*, prove that the triangle *ABG* is equivalent to one fourth of the parallelogram.

617. To divide a given triangle into two equivalent parts by drawing a straight line parallel to one of the sides.

618. To divide a given triangle into two equivalent parts by drawing a straight line perpendicular to one of the sides.

619. To divide a given trapezoid into two equivalent parts by drawing a straight line parallel to the bases.

620. To divide a given triangle into three equivalent parts by drawing straight lines through a given point in one of its sides.

621. To divide a given quadrilateral into two equivalent parts by drawing a straight line through a given vertex.

622. In the triangle *ABC*, *D* is the middle point of *BC*, *E* is the middle point of *AD*, *F* is the middle point of *BE*, and *G* is the middle point of *CF*. Prove that the triangle *EFG* is equivalent to one eighth of the triangle *ABC*.

623. In the parallelogram *ABCD*, *E* and *F* are the middle points of *AB* and *CD* respectively; *AF* and *CE* are drawn intersecting *BD* at *G* and *H* respectively, and *BF* and *DE* are drawn intersecting *AC* at *K* and *M* respectively. Prove that *GKHM* is a parallelogram equivalent to one ninth of the parallelogram *ABCD*.

624. If from the point D, in the base BC of the triangle ABC, straight lines are drawn parallel to the sides AB and AC, meeting AB at E and AC at F, prove that the triangle AEF is a mean proportional between the triangles BED and DFC.

625. D is the middle point of the base BC of the triangle ABC; BE is drawn, bisecting AD and meeting AC at E. Prove that the triangle BEC is double the triangle ABE.

626. If three straight lines AD, BE, and CF, drawn from the vertices of the triangle ABC to the opposite sides intersect at a common point G within the triangle, prove that $\frac{GD}{AD}+\frac{GE}{BE}+\frac{GF}{CF}=1$.

PROBLEMS OF COMPUTATION.

1. Find the ratio of a rectangular field 20 rd. long and 14 rd. wide to a rectangular field 18 rd. long and 16 rd. wide.

2. A rectangular field 48 rd. long contains 9 acres. Find the width.

3. The diagonal of a rectangle is 130, and the altitude is 32. Find the area.

4. A rectangle is 160 ft. long and 30 ft. wide. If the length is made 40 ft. less, how much must the breadth be increased in order that the area may remain the same?

5. The area of a rectangle is 104 sq. ft., and its perimeter is 42 ft. Find the length and breadth.

6. If the perimeter of a rectangle is 96 ft., and the length is three times the breadth, find the area.

7. If the area of a rectangle is 256 sq. ft., and the length is four times the breadth, find the length and breadth.

8. A rectangular garden 56 ft. long and 40 ft. wide has a path 6 ft. wide around it. Find the area of the path.

9. Find the ratio of the side of a square to its diagonal.

10. The area of the rhombus is 60 sq. ft., and one of its diagonals is 15 ft. Find the perimeter.

11. The area of a rhombus is 240, and its side is 17. Find the diagonals.

12. The sum of the diagonals of a rhombus is 32, and one diagonal is three fifths of the other. Find the area.

13. One angle of a rhombus is 60°, and a side is 8 ft. Find the area.

14. A board $16\frac{1}{2}$ ft. long is 19 in. wide at one end and 11 in. wide at the other end. Find the area.

15. The area of a trapezoid is 374, the altitude is 17, and one of the bases is 28. Find the other base.

16. The perimeter of a trapezoid is 56 in. If each of the legs is 13 in. long, and the area is 180 sq. in., find the lengths of the bases.

17. The bases of a trapezoid are 38 and 14, and each leg is 13. Find the area.

18. The bases of a trapezoid are $46\frac{1}{2}$ and 19, and the legs are $25\frac{1}{2}$ and 13. Find the area.

19. Find the side of a square equivalent to a trapezoid whose bases are 52 ft. and 60 ft., and whose altitude is 14 ft.

20. A diagonal of a trapezium is 31 ft., and the perpendiculars drawn to it from the opposite vertices are $8\frac{1}{2}$ ft. and 22 ft. Find the area.

21. The hypotenuse of a right triangle is 25 ft., and one leg is 7 ft. Find the area.

22. The area of a right triangle is 180 sq. ft., and the hypotenuse is 41 ft. Find the legs.

23. The segments of the hypotenuse of a right triangle, formed by the perpendicular drawn from the vertex of the right angle, are $9\frac{1}{3}$ and 21. Find the area of the triangle.

24. The sides of a right triangle are to each other as 8, 15, and 17, and the length of the perpendicular drawn from the vertex of the right angle to the hypotenuse is 30 ft. Find the area of the triangle.

25. ABC is a right triangle. The legs AC and BC are respectively 50 and 120 ft. If the triangle is divided into two equivalent parts by a line DF parallel to BC, find the lengths of the three sides of the triangle ADF.

26. Find the area of an isosceles right triangle whose hypotenuse is 6.

27. Find the hypotenuse of an isosceles right triangle whose area is $42\frac{1}{4}$.

28. In an isosceles right triangle, whose hypotenuse is 8, lines are drawn joining the middle points of the three sides. Find the area of each of the triangles thus formed.

29. In an isosceles triangle the base is 10 and either leg is 13. Find the area.

30. The altitude of an isosceles triangle is 6, and its perimeter is 36. Find the area.

31. The area of an isosceles triangle is 50 sq. ft., and each of its legs is 10 ft. Find the angles of the triangle.

32. Find the area of an equilateral triangle whose side is 3 ft.

33. Find the area of an equilateral triangle whose altitude is 6 ft.

34. If in two similar triangles a side of one is three times as long as the homologous side of the other, what is the ratio of the areas?

35. Two triangles have sides a, b, and c and a', b', and c'; the angles opposite a and a' are equal. Find the ratio of their areas.

36. Find the area of a triangle whose sides are 9 ft., 11 ft., and 12 ft.

37. The perimeter of a triangle is 29 ft., and the radius of the inscribed circle is 6 ft. Find the area of the triangle.

38. Two sides of a triangle are 10 and 15, and the area is 45. Find the third side.

39. The sides of a triangle are to each other as 3, 4, and 6, and the perimeter is $32\frac{1}{2}$; find the area.

40. The sides of a triangle are 13 ft., 37 ft., and 40 ft. Find the areas of the parts into which the triangle is divided by the bisector of the angle opposite the shortest side.

41. The area of a triangle is 240 sq. ft., and the base is 20 ft. If a line is drawn parallel to the base 6 ft. from the vertex of the triangle, find the area of the triangle thus formed.

42. In a triangle whose altitude is 6 ft., a line is drawn parallel to the base dividing it into two equivalent parts; find the altitude of the triangular part.

43. The base of a triangle is 14 in., and its altitude is 7 in. Find the area of the trapezoid cut off by a line drawn parallel to the base 6 in. from the vertex.

44. The area of a polygon is $122\frac{1}{2}$ sq. ft., and one side is 14 ft.; find the area of a similar polygon whose homologous side is 22 ft.

45. One side of a polygon is 28 ft., and the area is 384 sq. ft.; find the homologous side of a similar polygon whose area is 726 sq. ft.

46. If the areas of two similar polygons are as 16 to 25, and the perimeter of the first polygon is 50 in., what is the perimeter of the second?

47. The homologous sides of two similar polygons are 8 ft. and 15 ft.; find the homologous side of a similar polygon which is equivalent to their sum.

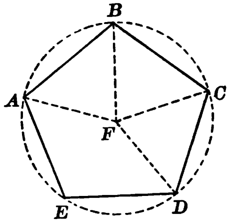
48. The area of a polygon is S square feet. Find the area of a similar polygon whose perimeter is in the ratio of m to n to that of the given polygon.

BOOK VI.

REGULAR POLYGONS. MEASUREMENT OF THE CIRCLE.

Proposition 191. Theorem.

227. *A circle can be circumscribed about any regular polygon.*



Hypothesis. The polygon $ABCDE$ is a regular polygon.

Conclusion. A $\odot$ can be circumscribed about $ABCDE$.

Proof. Describe a circumference through A, B, and C, and let F be its centre.

Draw FA, FB, FC, and FD.

$\angle ABC = \angle BCD$. (?)

$\angle FBC = \angle BCF$. (?)

$\therefore \angle ABF = \angle FCD$. (?)

$FB = FC$. (?)

$AB = CD$. (?)

$\therefore \triangle FAB = \triangle FCD$. (?)

$\therefore FA = FD$. (?)

$\therefore$ D is a point on the circumference. (?)

In like manner it may be proved that the circumference passes through the remaining vertices of the polygon.

∴ a ⊙ described with F as a centre and with FA as a radius will be circumscribed about the polygon $ABCDE$.

Q.E.D.

Proposition 192. Theorem.

228. *A circle can be inscribed in any regular polygon.*

Consult Prop. 191 and Prop. 107.

Cor. *The circles circumscribed about and inscribed in a regular polygon are concentric.*

229. The common centre of the circles circumscribed about and inscribed in a regular polygon is called the *centre* of the polygon. The radius of the circumscribed circle is called the *radius* of the polygon, and the radius of the inscribed circle is called the *apothem* of the polygon. An angle formed by two radii drawn to the extremities of a side of a regular polygon is called *an angle at the centre.*

In circles of unequal radii, arcs which subtend equal central angles are called *similar arcs.* Sectors whose bounding radii include equal central angles are called *similar sectors,* and segments whose bounding arcs are similar are called *similar segments.*

Proposition 193. Theorem.

230. *The angle at the centre of a regular polygon is equal to four right angles divided by the number of sides of the polygon.*

Ex. 627. The angle at the centre of a regular polygon is the supplement of the angle of the polygon.

Proposition 194. Theorem.

231. *Any angle of a regular polygon is bisected by the radius drawn to the vertex.*

Proposition 195. Theorem.

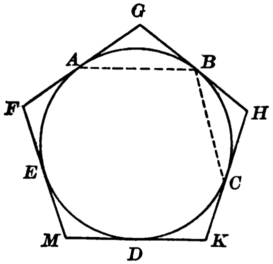
232. *If the circumference of a circle is divided into any number of equal arcs, the chords joining the successive points of division form an inscribed regular polygon.*

Consult Prop. 99 and Prop. 124, Cor. I.

COR. *If the vertices of an inscribed regular polygon are joined to the middle points of the arcs subtended by the sides of the polygon, an inscribed regular polygon of double the number of sides is formed.*

Proposition 196. Theorem.

233. *If the circumference of a circle is divided into any number of equal arcs, the tangents drawn at the points of division form a circumscribed regular polygon.*



Hypothesis. The circumference *ABD* is divided into equal parts at the points *A, B, C, D,* and *E,* and the polygon *FGHKM* is formed by tangents drawn at these points.

Conclusion. The polygon $FGHKM$ is a regular polygon.

Proof. Draw the chords AB and BC.

$$AB = BC. \qquad (?)$$

$$\angle GAB = \angle GBA = \angle HBC = \angle HCB. \qquad (?)$$

$$\therefore \triangle AGB = \triangle BHC, \text{ and the } \triangle\text{s are isosceles.} \qquad (?)$$

$$\angle G = \angle H. \qquad (?)$$

In like manner it may be proved that

$$\angle G = \angle F = \angle M, \text{ etc.}$$

$$AG = GB = BH = HC. \qquad (?)$$

In like manner it may be proved that

$$AG = AF = FE = EM, \text{ etc.}$$

$$\therefore MF = FG = GH, \text{ etc.} \qquad (?)$$

$$\therefore \text{the polygon } FGHKM \text{ is a regular polygon.} \qquad (?)$$

Q.E.D.

Cor. I. *Tangents to a circle at the vertices of an inscribed regular polygon form a circumscribed regular polygon of the same number of sides.*

Cor. II. *If tangents are drawn at the middle points of the arcs between adjacent points of contact of the sides of a circumscribed regular polygon, a circumscribed regular polygon of double the number of sides is formed.*

Ex. 628. An equilateral polygon inscribed in a circle is a regular polygon.

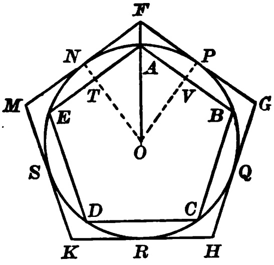
Ex. 629. An equiangular polygon circumscribed about a circle is a regular polygon.

Ex. 630. An equilateral polygon circumscribed about a circle is regular if it has an odd number of sides.

Ex. 631. An equiangular polygon inscribed in a circle is regular if it has an odd number of sides.

Proposition 197. Theorem.

234. *If a regular polygon is inscribed in a circle, the tangents drawn at the middle points of the arcs subtended by the sides of the polygon form a circumscribed regular polygon, whose sides are parallel to the sides of the inscribed polygon, and whose radii pass through the vertices of the inscribed polygon.*



Hypothesis. *ABCDE* is a regular polygon inscribed in the ⊙*ACD*, and *FGHKM* is a polygon formed by tangents drawn at *N, P, Q, R,* and *S*, the middle points of the arcs *AB, BC, CD, DE,* and *EA*, respectively.

Conclusion. *FGHKM* is a regular polygon; *FG*, any side of *FGHKM*, is ∥ to *AB*, the corresponding side of *ABCDE*; and *OF*, any radius of *FGHKM*, passes through *A*, a vertex of *ABCDE*.

Proof. Arc $EA =$ arc $AB =$ arc BC, etc. (?)

∴ Arc $EN =$ arc $NA =$ arc $AP =$ arc PB, etc. (?)

∴ Arc $NP =$ arc $PQ =$ arc QR, etc. (?)

∴ *FGHKM* is a circumscribed regular polygon. (?)

Draw *ON* and *OP*, intersecting *EA* and *AB* at *T* and *V* respectively.

OP is ⊥ to FG. (?)

OP is ⊥ to AB also. (?)

$\therefore$ FG is $\parallel$ to AB. (?)

$\angle NOF = \angle FOP$. (?)

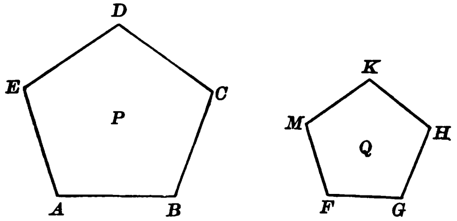
$AT = AV$. (?)

$\therefore$ A lies in OF. (?)

Q.E.D.

Proposition 198. Theorem.

235. *Two regular polygons of the same number of sides are similar.*



Hypothesis. P and Q are two regular polygons, each having n sides.

Conclusion. P and Q are similar polygons.

Proof. Each $\angle$ of each polygon $= \frac{2n-4}{n}$ rt. $\angle$s. (?)

$\therefore$ the polygons are mutually equiangular.

$AB = BC = CD$, etc., and $FG = GH = HK$, etc. (?)

$\therefore \frac{AB}{FG} = \frac{BC}{GH} = \frac{CD}{HK}$, etc. (?)

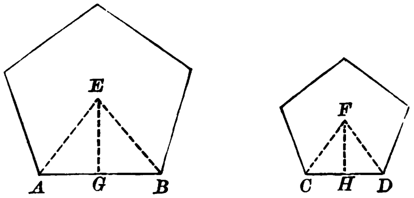
$\therefore$ P and Q are similar polygons. Q.E.D.

Cor. I. *The perimeters of two regular polygons of the same number of sides are to each other as any two homologous sides.*

Cor. II. *The areas of two regular polygons of the same number of sides are to each other as the squares of any two homologous sides.*

Proposition 199. Theorem.

236. *The perimeters of two regular polygons of the same number of sides are to each other as their radii, or as their apothems.*



Hint. Prove that ⊿ *ABE* and *CDF* are similar. Then consult Prop. 198, Cor. I., and Prop. 140.

Proposition 200. Theorem.

237. *The areas of two regular polygons of the same number of sides are to each other as the squares of their radii, or as the squares of their apothems.*

Use a method of proof similar to that used in Prop. 199.

Proposition 201. Theorem.

238. *The area of a regular polygon is equal to one half the product of its perimeter and apothem.*

Draw the radii of the polygon, and consult Prop. 166.

Proposition 202. Problem.

239. *To inscribe a square in a given circle.*

Hint. Draw two diameters perpendicular to each other, and join their extremities.

Proposition 203. Problem.

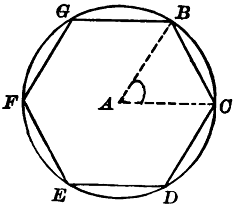
240. *To inscribe a regular octagon in a given circle.*

Consult Prop. 202 and Prop. 195, Cor.

Scholium. By continuing this process regular polygons may be inscribed of sixteen, thirty-two, sixty-four, etc. sides.

Proposition 204. Problem.

241. *To inscribe a regular hexagon in a given circle.*



To inscribe a regular hexagon in the ⊙BDF.

Construction. Draw any radius AB.

With B as a centre and with AB as a radius, describe an arc intersecting the circumference at C, and draw BC.

Then BC is a side of an inscribed regular hexagon, and the figure $BCDEFG$, formed by applying the radius six times as a chord, is an inscribed regular hexagon.

Proof. Draw AC.

$\triangle ABC$ is equilateral and equiangular. (?)

$\therefore \angle A = \frac{1}{3}$ of 2 rt. $\angle$s $= \frac{1}{6}$ of 4 rt. $\angle$s. (?)

$\therefore$ arc BC is $\frac{1}{6}$ of the circumference, and BC is a side of an inscribed regular hexagon. (?)

Q.E.F.

Cor. *The side of a regular hexagon is equal to its radius.*

Scholium. By joining the alternate vertices of an inscribed regular hexagon, an inscribed equilateral triangle is formed.

By proceeding as in Prop. 203, regular polygons may be inscribed of twelve, twenty-four, forty-eight, etc. sides.

Ex. 632. If r is the radius of a square, prove that a side of the square is equal to $r\sqrt{2}$, and that the apothem is equal to $\frac{1}{2}r\sqrt{2}$.

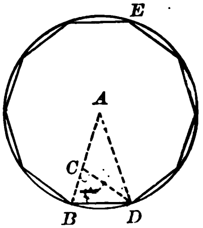
Ex. 633. If r is the radius of a regular hexagon, prove that the apothem of the hexagon is equal to $\frac{1}{2}r\sqrt{3}$.

Ex. 634. If r is the radius of an equilateral triangle, prove that a side of the triangle is equal to $r\sqrt{3}$, and that the apothem is equal to $\frac{1}{2}r$.

Ex. 635. The apothem of an inscribed regular hexagon is equal to one half the side of the inscribed equilateral triangle.

Proposition 205. Problem.

242. *To inscribe a regular decagon in a given circle.*



To inscribe a regular decagon in the ⊙ EDB.

Construction. Draw any radius AB, and divide it in extreme and mean ratio so that $AB : AC :: AC : CB$.

With B as a centre and with AC as a radius, describe an arc intersecting the circumference at D, and draw BD.

Then BD is a side of an inscribed regular decagon, and the figure formed by applying AC ten times as a chord is an inscribed regular decagon.

Proof. Draw AD and CD.

$AB : BD :: BD : CB.$ (?)

Then $\triangle$s ABD and CBD are similar. (?)

$\therefore$ $\triangle\, CBD$ is isosceles, and $CD = BD$. (?)

$\therefore$ $CD = AC$, and $\triangle\, ACD$ is isosceles. (?)

$\therefore \angle A = \angle ADC.$ (?)

But $\angle BCD = \angle A + \angle ADC.$ (?)

$\therefore \angle BCD = 2 \angle A.$ (?)

$\therefore \angle ABD = 2 \angle A$, and $\angle ADB = 2 \angle A$. (?)

$\angle A + \angle ABD + \angle ADB = 2$ rt. $\angle$s. (?)

$\therefore 5 \angle A = 2$ rt. $\angle$s. (?)

$\therefore \angle A = \frac{1}{5}$ of 2 rt. $\angle$s $= \frac{1}{10}$ of 4 rt. $\angle$s. (?)

$\therefore$ arc BD is $\frac{1}{10}$ of the circumference, and BD is a side of an inscribed regular decagon.

Q.E.F.

Cor. *If the radius of a circle is divided in extreme and mean ratio, the greater segment is equal to a side of an inscribed regular decagon.*

Scholium. By joining the alternate vertices of an inscribed regular decagon, an inscribed regular pentagon is formed.

By proceeding as in Prop. 203, regular polygons may be inscribed of twenty, forty, eighty, etc. sides.

Ex. 636. If r is the radius of a regular decagon, prove that a side of the decagon is equal to $\frac{1}{2} r(\sqrt{5} - 1)$, and that the apothem is equal to $\frac{1}{4} r\sqrt{10 + 2\sqrt{5}}$.

Ex. 637. An angle of a regular pentagon is trisected by the diagonals drawn from the vertex.

Ex. 638. The diagonals of a regular pentagon are equal.

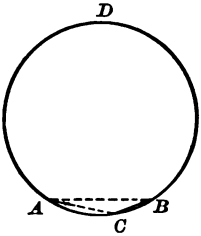
Ex. 639. Upon a given line to construct an isosceles triangle which shall have each of its base angles double the vertical angle.

Ex. 640. Upon a given line to construct an isosceles triangle which shall have its vertical angle three times either base angle.

Ex. 641. To divide a right angle into five equal parts.

Proposition 206. Problem.

243. *To inscribe a regular pentedecagon in a given circle.*



To inscribe a regular pentedecagon in the ⊙ DBA.

Construction. Draw AB, a side of an inscribed regular hexagon. Also draw AC, a side of an inscribed regular decagon, and draw CB.

Then CB is a side of an inscribed regular pentedecagon, and the figure formed by applying CB fifteen times as a chord is an inscribed regular pentedecagon.

Proof. The arc AB is $\frac{1}{6}$ of the circumference, and the arc AC is $\frac{1}{10}$ of the circumference. (?)

$\therefore$ the arc CB is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$, of the circumference.

Then CB is a side of an inscribed regular pentedecagon.

Q.E.F.

SCHOLIUM. By proceeding as in Prop. 203, regular polygons may be inscribed of thirty, sixty, one hundred and twenty, etc. sides.

Ex. 642. Each diagonal of a regular pentagon is parallel to one of the sides.

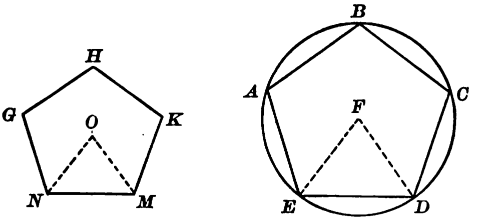
Ex. 643. If two diagonals of a regular pentagon intersect each other, the longer segments are each equal to a side of the pentagon.

Ex. 644. To construct a regular pentagon each side of which shall be equal to a given line.

Ex. 645. To construct a regular hexagon each side of which shall be equal to a given line.

Proposition 207. Problem.

244. *To inscribe in a given circle a regular polygon similar to a given regular polygon.*



To inscribe in the $\odot ACD$ a regular polygon similar to $GHKMN$, a regular polygon of n sides.

Construction. In the regular polygon $GHKMN$, draw the radii ON and OM.

From F, the centre of the $\odot$, draw FE and FD, making $\angle EFD = \angle NOM$.

Draw ED.

Then ED is a side of the required polygon, and the figure formed by applying ED n times as a chord is the required polygon.

[The proof is left to the student.]

Proposition 208. Problem.

245. *To circumscribe about a given circle a polygon similar to a given inscribed regular polygon.*

Consult Prop. 196, Cor. I., or Prop. 197.

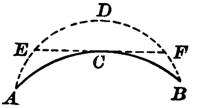
Ex. 646. In the figure of Prop. 205, prove that AC is a side of a regular pentagon inscribed in the circle circumscribed about the triangle ACD.

Proposition 209. Theorem.

246. *The circumference of a circle is greater than the perimeter of any inscribed polygon.*

Proposition 210. Theorem.

247. *An arc of a circle is less than any line which envelops it and has the same extremities.*



Hypothesis. ACB is an arc of a ⊙.

Conclusion. ACB is shorter than any line which envelops it and has the same extremities.

Proof. Taking together ACB and all the lines which envelop it and have the same extremities, there must be one line which is shorter than all the others or else is one of a number of equal lines which are shorter than the others. Suppose that ADB is such a line.

Through any point of the arc ACB draw the tangent EF.

$$\text{Then } AE + EF + FB < ADB. \qquad (?)$$

$\therefore$ ABD cannot be the shortest one of the lines mentioned. In like manner it may be proved that no one of the lines enveloping ACB can be the shortest line.

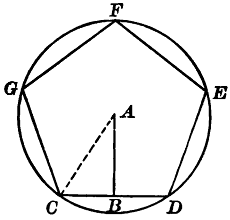
$\therefore$ ACB must be the shortest line, and ACB is less than any of the lines which envelop it and have the same extremities.

Q.E.D.

COR. *The circumference of a circle is less than the perimeter of any circumscribed polygon.*

Proposition 211. Theorem.

248. *If the number of the sides of an inscribed regular polygon is increased indefinitely, the apothem of the polygon approaches the radius of the circle as its limit.*



Hypothesis. AB is the apothem of the regular polygon inscribed in the ⊙ FDC.

Conclusion. The limit of AB, as the number of the sides is increased indefinitely, is the radius of the ⊙.

Proof. Draw the radius AC.

If the number of the sides is increased indefinitely, the length of each side is decreased indefinitely.

Then the length of CD can be made as small as we please.

Now $CB = \frac{1}{2} CD$. (?)

But $AC - AB < CB$. (?)

$\therefore$ $AC - AB$ can be made as small as we please, and the limit of $AC - AB$ is 0.

$\therefore$ lim. $AB = AC$.

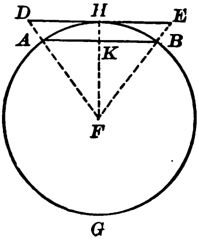
Q.E.D.

Cor. *The apothem of a regular polygon is the perpendicular bisector of the side to which it is drawn.*

Ex. 647. The perpendicular bisectors of the sides of a regular polygon pass through the centre.

Proposition 212. Theorem.

249. *If the number of sides of an inscribed or a circumscribed regular polygon is increased indefinitely, the perimeter of the polygon approaches the circumference of the circle as its limit.*



Hypothesis. Let p and P denote the perimeters of two regular polygons of the same number of sides, the former inscribed in the ⊙ ABG, and the latter circumscribed about the same ⊙. Let C denote the circumference of the ⊙, R the radius of the inscribed polygon, and r the apothem of the inscribed polygon.

Conclusion. As the number of the sides of the polygons is indefinitely increased, the limit of either p or P is C.

Proof. Let AB be the side of the inscribed polygon.

Draw the radius FH to the middle point of the arc AB, and draw DE tangent to the ⊙ at H. Draw the radii FA and FB, and produce them to meet DE at D and E.

Then DE is a side of the circumscribed polygon. (?)

$$\frac{P}{p}=\frac{R}{r}. \qquad (?)$$

$$\frac{P-p}{P}=\frac{R-r}{R}. \qquad (?)$$

$$\therefore\ P-p=\frac{P}{R}(R-r). \qquad (?)$$

If the number of the sides of each polygon is increased indefinitely, $R - r$ will approach 0 as a limit. (?)

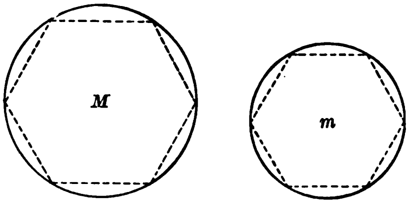
Hence $\frac{P}{R}(R - r)$, or $P - p$, will approach 0 as a limit.

But C has a value between P and p. (?)
Hence $P - C$ and $C - p$ will each approach 0 as a limit.
$\therefore$ P and p will each approach C as a limit.

Q.E.D.

Proposition 213. Theorem.

250. *The circumferences of two circles are to each other as their radii.*



Hypothesis. M and m are two Ⓢ, whose respective circumferences are C and c, and whose radii are R and r.

Conclusion. $\frac{C}{c} = \frac{R}{r}$.

Proof. Inscribe in the Ⓢ regular polygons of the same number of sides, and denote their perimeters by P and p.

Then $\frac{P}{p} = \frac{R}{r}$. (?)

Now let the number of the sides of each polygon be increased indefinitely, always keeping the same number of sides in the two polygons. Then P and p approach C and c as their respective limits. (?)

$\therefore \frac{C}{c} = \frac{R}{r}$. (?)

Q.E.D.

Cor. I. *Similar arcs are to each other as their radii.*

Cor. II. *The circumferences of two circles are to each other as their diameters.*

Cor. III. *The ratio of the circumference of a circle to its diameter is a constant quantity.*

Scholium. The constant ratio $\frac{C}{D}$, where C and D represent the circumference and diameter respectively of any circle, is denoted by the Greek letter π. It can be proved by the higher mathematics that π is incommensurable, and accordingly its numerical value can be expressed only approximately.

$$\text{Since } \frac{C}{D}=\pi,\ C=\pi D, \text{ or } C=2\pi R.$$

Ex. 648. To construct a circumference which shall be equal to the sum of two given circumferences.

Ex. 649. To construct a circumference which shall be equal to the difference of two given circumferences.

Ex. 650. To construct a circumference whose ratio to a given circumference shall equal the ratio of two given lines.

Ex. 651. In two circles of different radii, angles at the centre which intercept equal arcs are to each other inversely as the radii.

Proposition 214. Theorem.

251. *If the number of sides of an inscribed or a circumscribed regular polygon is increased indefinitely, its area approaches the area of the circle as its limit.*

Let A denote the area of the circle, s the area of the inscribed polygon, and S the area of the circumscribed polygon. Use a method of proof similar to that used in Prop. 212. In the figure there given, note that $R^2-r^2=\overline{AK}^2$.

Proposition 215. Theorem.

252. *The areas of two circles are to each other as the squares of their radii.*

Use a method of proof similar to that used in Prop. 213.

COR. *Similar sectors are to each other as the squares of their radii.*

Ex. 652. To construct a circle which shall be equivalent to the sum of two given circles.

Ex. 653. To construct a circle which shall be equivalent to the difference of two given circles.

Ex. 654. To construct a circle whose ratio to a given circle shall equal the ratio of two given lines.

Proposition 216. Theorem.

253. *Similar segments are to each other as the squares of their radii.*

Consult Prop. 215, Cor., Prop. 177, and §§ 95 and 98.

Ex. 655. The area of a circumscribed square is equal to twice the area of the inscribed square.

Ex. 656. The area of a circumscribed equilateral triangle is equal to four times the area of the inscribed equilateral triangle.

Ex. 657. The area of an inscribed regular hexagon is equal to three fourths the area of the circumscribed regular hexagon.

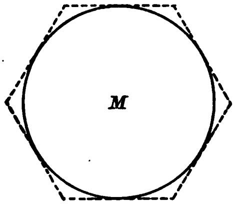
Ex. 658. The area of an inscribed equilateral triangle is equal to one half the area of the inscribed regular hexagon.

Ex. 659. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

Ex. 660. If the sides of a regular hexagon are produced, the star-shaped figure thus formed is equivalent to twice the hexagon.

Proposition 217. Theorem.

254. *The area of a circle is equal to one half the product of its circumference and radius.*



Hypothesis. Let A denote the area, C the circumference, and R the radius of the $\odot$ M.

Conclusion. Area of $M = \frac{1}{2} C \times R$.

Proof. Circumscribe a regular polygon about the $\odot$, and let S denote its area, and P its perimeter.

Then $$S = \tfrac{1}{2} P \times R. \qquad (?)$$

[To be completed by the student, using the method of limits.]

Cor. I. *The area of a sector is equal to one half the product of its arc and radius.*

Cor. II. *The area of a circle is equal to π times the square of its radius.*

$$A = \tfrac{1}{2} C \times R = \tfrac{1}{2} \times 2\pi R \times R = \pi R^2.$$

Cor. III. *The area of a circle is equal to $\frac{1}{4}\pi$ times the square of its diameter.*

$$A = \pi R^2 = \pi \left(\frac{D}{2}\right)^2 = \tfrac{1}{4}\pi D^2.$$

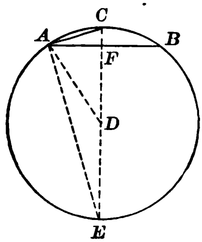
Ex. 661. The area of a circle is equal to four times the area of the circle described on its radius as a diameter.

Ex. 662. The area of a circular ring is equal to the area of a circle whose diameter is a chord of the outer circle which is tangent to the *inner circle.*

Ex. 663. If three circles are described on the sides of a right triangle as diameters, the circle described on the hypotenuse is equivalent to the sum of the circles described on the legs.

Proposition 218. Problem.

255. *Given the radius and the side of an inscribed regular polygon, to compute the side of the inscribed regular polygon of double the number of sides.*



Let a denote the side of a regular polygon inscribed in the ⊙ whose radius is r, and let a' denote the side of the inscribed regular polygon of double the number of sides.

To compute the value of a'.

Let AB be the side a, and let AC be the side a'.

Through D, the centre of the ⊙, draw the diameter CE intersecting AB at F. Also draw AD and AE.

CE is the ⊥ bisector of AB. (?)

$$\overline{FD}^2 = \overline{AD}^2 - \overline{AF}^2 = r^2 - \tfrac{1}{4}a^2. \qquad \text{(?)}$$

$$\therefore\ FD = \sqrt{r^2 - \tfrac{1}{4}a^2} = \tfrac{1}{2}\sqrt{4r^2 - a^2}. \qquad \text{(?)}$$

$$a'^2 = \overline{AC}^2 = CE \times CF = 2r(r - FD). \qquad \text{(?)}$$

$$\therefore\ a'^2 = 2r\left(r - \tfrac{1}{2}\sqrt{4r^2 - a^2}\right) = 2r^2 - r\sqrt{4r^2 - a^2}. \qquad \text{(?)}$$

$$\therefore\ a' = \sqrt{2r^2 - r\sqrt{4r^2 - a^2}}. \qquad \text{(?)}$$

Q.E.F.

Scholium. If the diameter of the circle equals unity, the formula becomes

$$a' = \sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - a^2}}.$$

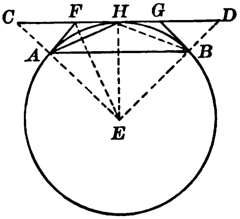
Ex. 664. If r is the radius of a regular pentagon, prove that a side of the pentagon is equal to $\frac{1}{2}r\sqrt{10-2\sqrt{5}}$, and that the apothem is equal to $\frac{1}{4}r(1+\sqrt{5})$.

Ex. 665. If r is the radius of a regular octagon, prove that a side of the octagon is equal to $r\sqrt{2-\sqrt{2}}$, and that the apothem is equal to $\frac{1}{2}r\sqrt{2+\sqrt{2}}$.

Ex. 666. If r is the radius of a regular dodecagon, prove that a side of the dodecagon is equal to $r\sqrt{2-\sqrt{3}}$, and that the apothem is equal to $\frac{1}{2}r\sqrt{2+\sqrt{3}}$.

Proposition 219. Problem.

256. *Given the perimeters of inscribed and circumscribed regular polygons of the same number of sides, to compute the perimeters of inscribed and circumscribed regular polygons of double the number of sides.*



Let p and P denote respectively the perimeters of two inscribed and circumscribed polygons of n sides each, and let p' and P' denote respectively the perimeters of two inscribed and circumscribed polygons of $2n$ sides each.

To compute the values of p' and P'.

Let AB be a side of the polygon whose perimeter is p.

Draw the radius EH to the middle point of the arc AB, and draw CD tangent to the ⊙ at H. Draw the radii EA and EB, and produce them to meet CD at C and D.

Then CD is a side of the polygon whose perimeter is P. (?)

Draw AH, and at A and B draw the tangents AF and BG, meeting AB at F and G.

Then AH is a side of the polygon whose perimeter is p', and FG is a side of the polygon whose perimeter is P'. (?)

Draw EF and BH.

$$\frac{P}{p}=\frac{EC}{EH}. \qquad (?)$$

$$\text{But } \frac{EC}{EH}=\frac{CF}{FH}. \qquad (?)$$

$$\therefore \frac{P}{p}=\frac{CF}{FH}. \qquad (?)$$

$$\therefore \frac{P+p}{p}=\frac{CF+FH}{FH}=\frac{CH}{FH}. \qquad (?)$$

$$\therefore \frac{P+p}{p}=\frac{CD}{FG}. \qquad (?)$$

$$\text{Now } CD=\frac{P}{n}, \text{ and } FG=\frac{P'}{2n}. \qquad (?)$$

$$\therefore \frac{P+p}{p}=\frac{\frac{P}{n}}{\frac{P'}{2n}}=\frac{2P}{P'}. \qquad (?)$$

$$\therefore P'(P+p)=2Pp. \qquad (?)$$

$$\therefore P'=\frac{2Pp}{P+p}. \qquad (?)$$

$\triangle$ HAB and FAH are similar. (?)

$$\therefore \frac{AB}{AH}=\frac{AH}{FH}. \qquad (?)$$

$$\therefore \overline{AH}^2=AB\times FH. \qquad (?)$$

$$\text{Now } AH=\frac{p'}{2n},\ AB=\frac{p}{n}, \text{ and } FH=\frac{P'}{4n}. \qquad (?)$$

$$\therefore \frac{p'^2}{4n^2}=\frac{p}{n}\times\frac{P'}{4n}. \qquad (?)$$

$$\therefore p'^2=pP'. \qquad (?)$$

$$\therefore p'=\sqrt{pP'}. \qquad (?)$$

Q.E.F.

Proposition 220. Problem.

257. *To compute the numerical value of π approximately.*

First Method.

In the circle whose diameter equals unity, the formula $C = 2\pi R$ becomes $C = \pi$.

The side of a regular hexagon inscribed in this circle equals $\frac{1}{2}$, and its perimeter equals 3.

The side of the inscribed regular dodecagon equals

$$\sqrt{\tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - \tfrac{1}{4}}} = 0.258819.$$

The perimeter of the inscribed regular dodecagon equals $0.258819 \times 12 = 3.105829$.

By continuing this process, the following table is formed:

No. of Sides.	Length of Side.	Length of Perimeter.
6	0.500000	3.000000
12	0.258819	3.105829
24	0.130526	3.132629
48	0.065403	3.139350
96	0.032719	3.141032
192	0.016362	3.141452
384	0.008181	3.141558
768	0.004091	3.141585
1536	0.002045	3.141589

Each number in the last column is an approximation to the numerical value of π, and the greater the number of the sides of the polygon, the greater the degree of accuracy. The last two numbers agree to the fifth decimal place. Hence an approximate value of π is 3.14159, which is correct to the fifth decimal place.

Second Method.

The side of a circumscribed square is equal to the diameter of the circle, and the diagonal of an inscribed square is also equal to the diameter of the circle. Hence in the circle whose diameter equals unity, the perimeter of the circumscribed square equals 4, and the perimeter of the inscribed square equals $4\sqrt{\frac{1}{2}} = 2.828427$. Denote these perimeters by P and p, respectively, and let P' and p' denote respectively the perimeters of circumscribed and inscribed octagons.

$$\text{Then } P' = \frac{2\,Pp}{P+p} = \frac{2 \times 4 \times 2.828427}{4 + 2.828427} = 3.313709,$$

$$\text{and } \quad p' = \sqrt{pP'} = \sqrt{2.828427 \times 3.313709} = 3.061467.$$

By continuing this process, the following table is formed:

No. of Sides.	Length of Perimeter of Circumscribed Polygon.	Length of Perimeter of Inscribed Polygon.
4	4.000000	2.828427
8	3.313709	3.061467
16	3.182598	3.121445
32	3.151725	3.136549
64	3.144118	3.140331
128	3.142224	3.141277
256	3.141750	3.141514
512	3.141632	3.141573
1024	3.141603	3.141588
2048	3.141595	3.141591

Since the circumference of the circle is less than the perimeter of the circumscribed polygon, and is greater than the perimeter of the inscribed polygon, the value of π must be less than 3.141595, and greater than 3.141591. These two numbers agree to the fifth decimal place. Hence an approximate value of π is 3.14159, which is correct to the fifth decimal place. Q.E.F.

SCHOLIUM. $\pi = 3\frac{1}{7}$ is sufficiently accurate for many purposes, and this approximate value is largely used in practical computations.

EXERCISES.

667. To inscribe a circle in a given square.

668. To circumscribe a circle about a given square.

669. Two diagonals of a regular pentagon, not drawn from the same vertex, divide each other in extreme and mean ratio.

670. The figure bounded by the five diagonals of a regular pentagon is a regular pentagon.

671. In the regular pentagon $ABCDE$. the diagonals AC and BD intersect at F; prove that $AC = AB + CF$.

672. The apothem of an inscribed regular pentagon is equal to one half the sum of the radius of the circle and the side of the inscribed regular decagon.

673. If AB is a side of an equilateral triangle inscribed in the circle of which C is the centre and ACD a diameter, prove that the triangle BCD is equilateral.

674. If D is the point of intersection of the altitudes of the equilateral triangle ABC, prove that a regular hexagon is formed by joining the middle points of AD, BD, and CD to the middle points of the nearest sides.

675. If two chords of a circle are perpendicular to each other, the sum of the four circles described on the four segments as diameters is equivalent to the given circle.

676. If the diameter of a circle is divided into two parts, and upon these parts semi-circumferences are described on opposite sides of the diameter, the circle is thus divided into two parts which have the same ratio as the two parts of the diameter.

677. To divide a given circle into two equivalent parts by a concentric circumference.

678. The radius of an inscribed regular polygon is a mean proportional between its apothem and the radius of the circumscribed regular polygon of the same number of sides.

679. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

680. The square of a side of an inscribed equilateral triangle is equal to three times the square of a side of the inscribed regular hexagon.

681. The square of the side of an inscribed equilateral triangle is equal to the sum of the squares of the sides of the inscribed square and of the inscribed regular hexagon.

682. The square of the side of an inscribed regular pentagon is equal to the sum of the squares of the radius of the circle and of the side of the inscribed regular decagon.

683. In a given equilateral triangle, to inscribe three equal circles which shall touch each other and also touch the sides of the triangle.

684. In a given circle, to inscribe three equal circles which shall touch each other and also touch the given circle.

685. In a given square, to inscribe four equal circles which shall touch each other, each circle to touch one side of the square..

686. In a given square, to inscribe four equal circles which shall touch each other, each circle to touch two sides of the square.

687. In a given equilateral triangle, to inscribe six equal circles which shall touch each other and also touch the sides of the triangle.

688. To inscribe a square in a given isosceles right triangle.

689. To inscribe an equilateral triangle in a given regular pentagon.

690. To inscribe a square in a given regular pentagon.

691. To inscribe a square in a given regular hexagon.

692. To inscribe a regular hexagon in a given equilateral triangle.

693. To inscribe a regular octagon in a given square.

694. The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the polygon is equal to the apothem multiplied by the number of its sides.

695. If squares are constructed outwardly upon the sides of a regular hexagon, the exterior vertices of these squares are the vertices of a regular dodecagon.

696. The figure formed by the straight lines joining the alternate vertices of a regular hexagon is a regular hexagon.

697. The area of an inscribed regular octagon is equal to the area of a rectangle whose sides are equal to the sides of the inscribed and circumscribed squares.

698. The area of an inscribed dodecagon is equal to three times the square of its radius.

PROBLEMS OF COMPUTATION.

[Use for π the approximate value $3\frac{1}{7}$.]

1. If the diameter of a circle is 3 in., what is the length of an arc of 80°?

2. If an arc of a circle is equal in length to the radius, what is the value of the subtended central angle?

3. The area of a circle is 154 sq. in.; find the value of the central angle subtended by an arc $5\frac{1}{2}$ in. long.

4. The radii of two circles are 2 ft. and 4 ft.; the area of the second is how many times the area of the first?

5. The radius of a given circle is 10 in.; what is the radius of a circle having twice the area of the given circle? Of a circle having one half the area of the given circle?

6. A sector whose angle is 40° has an area of 154 sq. ft.; find the perimeter of the sector.

7. The area of a regular decagon circumscribed about a given circle is 10 sq. in.; find the area of a similar figure circumscribed about a circle whose radius is 16 times as great as that of the first.

8. Find the length of the arc subtended by one side of a regular decagon inscribed in a circle whose radius is $10\frac{1}{2}$ ft.

9. The length of the circumference of a circle is 132 ft. Calculate the length of the diameter, the length of an arc of 40°, and the area of a sector of 80°.

10. The diameters of two concentric circles are 10 ft. and 6 ft.; find the area of the ring contained between their circumferences.

11. The circumferences of two concentric circles are 120 ft. and 80 ft.; find the width of the ring between them.

12. The angular measures of two arcs of the same length are 25° and 40°; if the radius of the first arc is 12 ft., what is the radius of the second?

13. Regular hexagons are inscribed in and circumscribed about a circle. If the apothem of the circumscribed hexagon is 12, what is the apothem of the inscribed hexagon?

14. Compute the area of a regular hexagon whose side is 5 ft.

15. A regular hexagon, the perimeter of which is 42 in., is inscribed in a circle; what is the area of this circle?

16. If the circumference of a circle is $37\frac{5}{7}$, find the perimeter of a circumscribed regular hexagon.

17. The area of a regular hexagon inscribed in a circle is $24\sqrt{3}$; find the area of the circle and the length of its circumference.

18. The length of a side of a given regular hexagon is 6; find the circumference and the area of the circle inscribed in a regular hexagon whose area is four times as great.

19. The length of the radius of a circle is 12 ft.; what are the lengths of the sides of the inscribed and circumscribed hexagons? What is the area of that portion of the circle contained between the circumference and the sides of the inscribed hexagon?

20. If the apothem of a regular hexagon is 2, find the area of the circumscribed circle.

21. The perimeter of a regular hexagon is 18; find the area of the circumscribed circle and the area of the square inscribed in this circle.

22. The area of a square is 49 sq. ft.; find the length of the circumference and the area of the circle inscribed in this square.

23. Find the area of the largest square that can be cut from a circle 16 in. in diameter.

24. Find the area of the circle in which a square, each side of which is $\sqrt{8}$ in. long, can be inscribed; also find the radius of a second circle which shall be nine times as large as the first.

25. Find the area included between the circumference of a circle of radius 7 and the square inscribed within it.

26. A certain equilateral triangle has sides $8\sqrt{3}$ in. long; what is the radius of the circle circumscribed about this triangle?

27. Find the radius and the area of the largest circle that can be cut from a triangle, every side of which is 4 ft. long.

28. The altitude of an equilateral triangle is 9 ft.; find the radii and the areas of the inscribed and circumscribed circles.

29. The area of an equilateral triangle is $36\sqrt{3}$; find the area of the circumscribed circle.

30. The area of a circle is 154 sq. ft.; find the area of the inscribed regular hexagon and of the inscribed equilateral triangle.

31. A regular hexagon *ABCDEF* is inscribed in a circle whose radius is 2; find the length of the diagonal *AC*.

32. Each side of an equilateral triangle measures 4 in. Find the

area contained between its perimeter and the circumference of the inscribed circle.

33. Find the area of a circle in which two chords drawn from the same point in the circumference to the extremities of a diameter measure 20 in. and 21 in.

34. In a circle of 4 in. radius, a chord is drawn perpendicular to a radius at its middle point; find the areas of the segments thus formed.

35. In a circle of 5 in. radius, a chord is drawn equal in length to the radius. What angle does it subtend at the centre of the circle? Compute the areas of the sector including this angle, and of the two segments into which the chord divides the circle.

36. Two circles are tangent internally, the ratio of their radii being $\frac{2}{3}$. What part of the larger circle is included between the two circumferences?

37. Four circles, each one foot in diameter, are drawn with their centres at the corners of a square whose side is one foot. Find the area of the innermost figure enclosed by arcs of the circles.

38. Letting *s* denote the length of the side of an equilateral triangle, find the radius of the circle whose area equals that of the triangle.

39. The radius of a circle is 9 in.; find the area of that one of all the regular polygons inscribed in it which has the shortest perimeter. How long a perimeter can a regular polygon inscribed in this circle have?

40. Three equal circles touch each other. Find the area of the three-pointed figure which they enclose, the radius of each circle being one foot.

41. Through a point without a circle whose radius is 105, two tangents are drawn, intercepting an arc whose length is 55; find the value of the angle formed by the tangents.

42. Through a point without a circle whose radius is 21 in., two tangents are drawn, forming an angle of 60°. Find the area of the figure bounded by the tangents and the arc intercepted between them.

43. A circular plot of ground containing $908\frac{2}{7}$ sq. rd. is surrounded by a path of uniform width, which contains 110 sq. rd.; find the width of the path.

44. Two flower-beds have equal perimeters. One of the beds is circular, and the other has the form of a regular hexagon. The circular

bed is surrounded by a walk 7 ft. wide bounded by a circumference concentric with the bed. The area of the walk is to that of the bed as 7 to 9. Find the diameter of the circular bed and the area of the hexagonal bed.

45. The perimeter of a certain church window is made up of three equal semi-circumferences, the centres of which are the vertices of an equilateral triangle whose sides are each $3\frac{1}{2}$ ft. long. Find the area of the window and the length of its perimeter.

46. A quarter-mile running track consists of two parallel straight portions joined together at the ends by semi-circumferences. The extreme length of the plot enclosed by the track is 180 yd. Find the cost of sodding this plot at 25 cts. per square yard.

47. A horse is tethered to a hook on the inner side of a fence which bounds a circular grass plot. His tether is so long that he can just reach the centre of the plot. The area of so much of the plot as he can graze over is $\frac{98}{3}(4\pi - 3\sqrt{3})$ sq. rd.; find the length of the tether and the circumference of the plot.

48. Every cross-section of the train house of a railway station has the form of a pointed arch made up of two circular arcs, the centres of which are on the ground. The radius of each arc is equal to the width of the building (210 ft.). Find the distance across the building measured over the roof and the area of the cross-section.

49. A stone bridge 20 ft. wide has a circular arch of 140 ft. span at the water level. The crown of the arch is $140(1 - \frac{1}{2}\sqrt{3})$ ft. above the surface of the water. How many square feet of surface must be gone over in cleaning so much of the arch as is above water?

50. A kite-shaped racing track is formed by a circular arc and two tangents at its extremities. The tangents meet at an angle of 60°. Two riders are to go around the track, one on a line close to the inner edge, the other on a line everywhere $5\frac{1}{4}$ ft. outside the first line. If the distance covered by the first rider is exactly a mile, how much over a mile does the second ride?

BOOK VII.

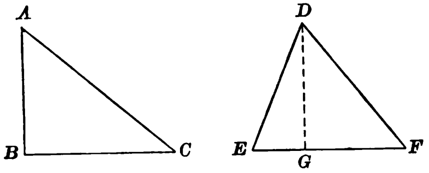
MAXIMA AND MINIMA. SYMMETRY.

258. In considering quantities of the same kind, that which has the greatest value is called the *maximum*, and that which has the smallest value is called the *minimum*.

Figures which have equal perimeters are called *isoperimetric* figures.

Proposition 221. Theorem.

259. *Of all triangles having two given sides, that in which these sides are perpendicular to each other is the maximum.*



Hypothesis. In the $\triangle ABC$ and DEF, $AB = DE$ and $BC = EF$; also AB is $\perp$ to BC.

Conclusion. $\triangle ABC > \triangle DEF$.

Proof. Draw $DG \perp$ to EF.

$$\frac{\triangle ABC}{\triangle DEF} = \frac{AB}{DG}. \qquad (?)$$

$$DE > DG. \qquad (?)$$

$$\text{But } AB = DE. \qquad (?)$$

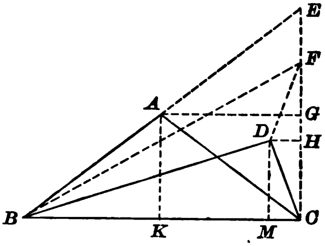
$$\therefore AB > DG. \qquad (?)$$

$$\therefore \triangle ABC > \triangle DEF. \qquad (?)$$

Q.E.D.

Proposition 222. Theorem.

260. *Of all isoperimetric triangles having the same base, that which is isosceles is the maximum.*



Hypothesis. ABC and DBC are two isoperimetric triangles, having the same base BC, and ABC is an isosceles $\triangle$, having $AB = AC$.

Conclusion. $\triangle ABC > \triangle DBC$.

Proof. Produce BA to E, making $AE = AB$, and draw CE.

Then $\angle BCE$ is a rt. $\angle$. (?)

Draw AG and $DH \perp$ to CE, and AK and $DM \perp$ to BC.

Take $HF = HC$, and draw BF and DF.

$$BE = BA + AE \qquad (?)$$
$$= BA + AC \qquad (?)$$
$$= BD + DC \qquad (?)$$
$$= BD + DF. \qquad (?)$$

Now $BD + DF > BF$. (?)

$\therefore BE > BF$. (?)

$\therefore EC > FC$. (?)

But $GC = \frac{1}{2} EC$, and $HC = \frac{1}{2} FC$. (?)

$\therefore GC > HC$. (?)

But $AK = GC$, and $DM = HC$. (?)

$\therefore AK > DM$. (?)

Now $\frac{\triangle ABC}{\triangle DBC} = \frac{AK}{DM}$. (?)

$\therefore \triangle ABC > \triangle DBC$. (?)

Q.E.D.

Ex. 699. A square has a greater area than any rhombus which has the same perimeter.

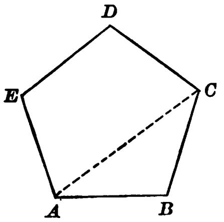
Ex. 700. Of all equivalent parallelograms having equal bases, the rectangle has the least perimeter.

Ex. 701. Of all equivalent triangles having the same base, that which is isosceles has the minimum perimeter.

Ex. 702. Of all equivalent parallelograms having the same diagonal, the rhombus has the minimum perimeter.

Proposition 223. Theorem.

261. *Of all isoperimetric polygons of the same number of sides, the maximum is equilateral.*



Hypothesis. $ABCDE$ is the maximum of all isoperimetric polygons of a given number of sides.

Conclusion. $AB = BC = CD$, etc.

Proof. Draw AC.

If AB and BC are unequal, an isosceles △ with an equal perimeter may be constructed on AC as a base, which will be greater than the △ ABC. (?)

The area of the polygon will thus be increased without changing the length of the perimeter, which is contrary to the hypothesis that the polygon is the maximum.

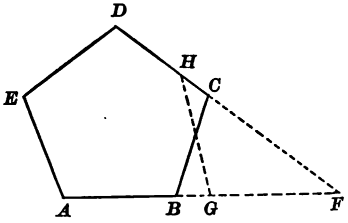
∴ AB and BC cannot be unequal, and $AB = BC$.

In like manner it may be proved that $BC = CD$, etc.

Q.E.D.

Proposition 224. Theorem.

262. *Of all isoperimetric equilateral polygons of the same number of sides, the maximum is equiangular.*



Hypothesis. $ABCDE$ is the maximum of all isoperimetric equilateral polygons of a given number of sides.

Conclusion. $\angle ABC = \angle BCD = \angle CDE$, etc.

Proof. Produce AB and DC till they meet at F.

If possible, suppose that $\angle ABC > \angle BCD$.

Then $\angle FBC < \angle FCB$, and $FC < FB$. (?)

Take $FG = FC$ and $FH = FB$, and draw GH.

Then $\triangle GFH = \triangle BFC$. (?)

$\therefore$ polygon $AGHDE \Leftrightarrow$ polygon $ABCDE$. (?)

$GH = BC$, and $AG + HD = AB + CD$. (?)

$\therefore$ polygons $AGHDE$ and $ABCDE$ are isoperimetric. (?)

$\therefore$ $AGHDE$ is the maximum of polygons having the given perimeter. (?)

$\therefore$ $AGHDE$ is equilateral, and $AG = AE$. (?)

But $AB = AE$. (?)

$\therefore$ $AG = AB$, which is impossible. (?)

$\therefore$ $\angle ABC$ cannot be greater than $\angle BCD$.

In like manner it may be proved that $\angle ABC$ cannot be less than $\angle BCD$.

$\therefore$ $\angle ABC = \angle BCD$.

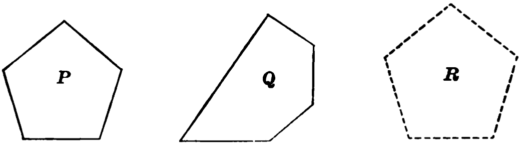
In like manner it may be proved that $\angle BCD = \angle CDE$, etc.

Q.E.D.

COR. *Of all isoperimetric polygons of the same number of sides, the maximum is regular.*

Proposition 225. Theorem.

263. *Of all equivalent polygons of the same number of sides, the regular polygon has the minimum perimeter.*



Hypothesis. P is a regular polygon of n sides, and Q is an irregular polygon having the same number of sides and the same area as P.

Conclusion. Perimeter of $Q >$ perimeter of P.

Proof. With a side equal to $\frac{1}{n}$ of the perimeter of Q, construct the regular polygon R similar to P.

Then $R > Q$. (?)

But $P \Leftrightarrow Q$. (?)

$\therefore R > P$. (?)

$\therefore$ perimeter of $R >$ perimeter of P. (?)

But perimeter of $Q =$ perimeter of R. (?)

$\therefore$ perimeter of $Q >$ perimeter of P. (?)

Q.E.D.

Ex. 703. Of all isoperimetric rectangles, the square is the maximum.

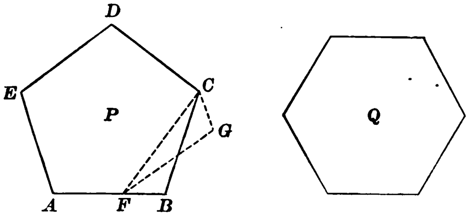
Ex. 704. Of all triangles having a given base and a given vertical angle, that which is isosceles is the maximum.

Ex. 705. Of all triangles having a given base and a given vertical angle, that which is isosceles has the maximum perimeter.

Ex. 706. Of all triangles having a given altitude and a given vertical angle, that which is isosceles is the minimum.

Proposition 226. Theorem.

264. *Of all isoperimetric regular polygons, that which has the greatest number of sides is the maximum.*



Hypothesis. P is a regular polygon of n sides, and Q is a regular polygon of $n+1$ sides, having a perimeter equal to that of P.

Conclusion. $Q > P$.

Proof. Draw CF from C to any point in AB. From C draw CG, making $\angle FCG = \angle CFB$, and from F draw FG, making $\angle CFG = \angle BCF$.

Then $\triangle FGC = \triangle FBC$. (?)

$\therefore$ polygons $AFGCDE$ and P are isoperimetric and equivalent. (?)

$\therefore$ polygons $AFGCDE$ and Q are isoperimetric. (?)

$\therefore$ $Q >$ polygon $AFGCDE$. (?)

$\therefore$ $Q > P$. (?)

Q.E.D.

Cor. *Of all isometric plane figures, the circle is the maximum.*

Ex. 707. Of all equivalent triangles having a given base, that which is isosceles has the maximum vertical angle.

Ex. 708. Of all rectangles that can be inscribed in a given circle, the square is the maximum.

Ex. 709. Of all triangles that can be inscribed in a given circle, that which is equilateral is the maximum, and has the maximum perimeter.

Proposition 227. Theorem.

265. *Of all equivalent regular polygons, that which has the greatest number of sides has the minimum perimeter.*

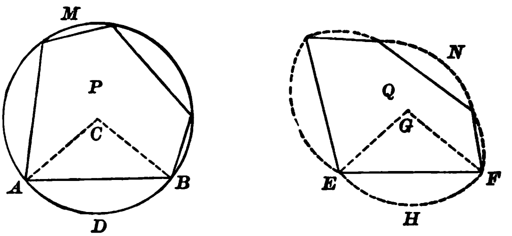
Let P be a regular polygon of n sides, and Q a regular polygon of more than n sides, having an area equal to that of P.

With a side equal to $\frac{1}{n}$ of the perimeter of Q construct a regular polygon similar to P, and proceed as in Prop. 225.

Cor. *Of all equivalent plane figures, the circle has the minimum perimeter.*

Proposition 228. Theorem.

266. *Of all mutually equilateral polygons, that which can be inscribed in a circle is the maximum.*



Hypothesis. P and Q are two mutually equilateral polygons, of which P is inscribed in the $\odot M$, while Q cannot be inscribed in a $\odot$.

Conclusion. $P > Q$.

Proof. Draw the radii CA and CB. On EF construct the $\triangle EFG = \triangle ABC$. With G as a centre and with GE as a radius, describe the arc EF.

Then arc $EF =$ arc AB. (?)

segment $EHF =$ segment ADB. (?)

In like manner construct segments on the other sides of Q equal respectively to the segments on the corresponding sides of P, and denote the irregular figure thus formed by N.

Then M and N are isoperimetric. (?)

$\therefore M > N$. (?)

$\therefore P > Q$. (?)

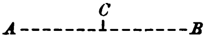
Q.E.D.

Ex. 710. If a straight line is divided into two parts, the product of the two segments is a maximum when the segments are equal.

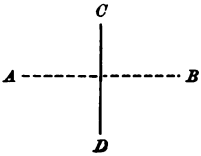
Ex. 711. If a straight line is divided into two parts, the sum of the squares of the parts is a minimum when the segments are equal.

SYMMETRY.

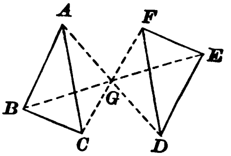
267. Two points are said to be *symmetrical* with respect to a third point, called the *centre of symmetry*, when this third point bisects the straight line which joins them. The distance of either of two symmetrical points from the centre of symmetry is called the *radius of symmetry*. For example, if the line AB is bisected at C, A and B are symmetrical with respect to C as a centre, and AC is the radius of symmetry.

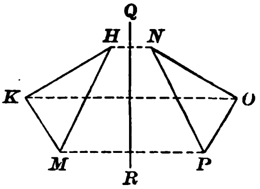


Two points are said to be *symmetrical* with respect to a straight line called the *axis of symmetry*, when this straight line is the perpendicular bisector of the straight line which joins them. For example, if the line CD is the perpendicular bisector of the line AB, A and B are symmetrical with respect to CD as an axis.

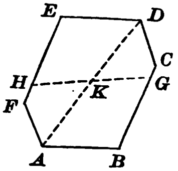


Two figures are said to be symmetrical with respect to a centre or an axis when every point of one figure has a corresponding symmetrical point in the other. For example, if every point in the figure *ABC* has a corresponding symmetrical point in *DEF* with respect to *G* as a centre, the figures *ABC* and *DEF* are symmetrical with respect to *G* as a centre. If every point in the figure *HKM* has a corresponding symmetrical point in *NOP* with respect to *QR* as an axis, the figures *HKM* and *NOP* are symmetrical with respect to *QR* as an axis.

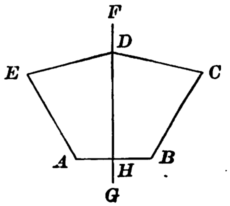




A figure is said to be symmetrical with respect to a centre when every straight line passing through the centre intersects the figure in two points which are symmetrical with respect to this centre. Any straight line passing through the centre of symmetry and having its extremities in the perimeter is called a *diameter of symmetry.* For example, the figure *ABCDEF* is symmetrical with respect to the centre *K*, as each line passing through *K* intersects the perimeter in two points which are symmetrical with respect to *K*; *AD* and *GH* are diameters of symmetry.

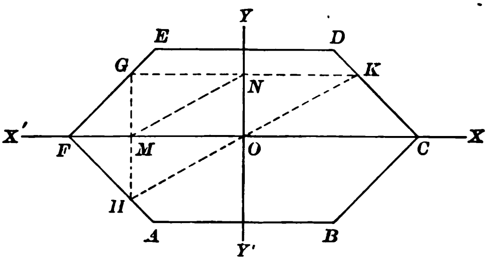


A figure is said to be symmetrical with respect to an axis when the axis divides the figure into two figures which are symmetrical with respect to that axis. For example, the figure *ABCDE* is symmetrical with respect to the axis *FG*, as the two figures *AHDE* and *BHDC* are symmetrical with respect to *FG*.



Proposition 229.

268. *If a figure is symmetrical with respect to two axes perpendicular to each other, it is also symmetrical with respect to their point of intersection as a centre.*



Hypothesis. The figure *ABCDEF* is symmetrical with respect to the axes *XX'* and *YY'*, which are ⊥ to each other and intersect at *O*.

Conclusion. The figure *ABCDEF* is symmetrical with respect to *O* as a centre.

Proof. Let *G* be any point in the perimeter.

Draw *GM* ⊥ to *XX'* and *GN* ⊥ to *YY'*, and produce these lines to meet the perimeter at *H* and *K* respectively.

Draw *HO*, *KO*, and *MN*.

$$GM = MH. \qquad (?)$$

$$GM = NO. \qquad (?)$$

$$\therefore\ MH = NO. \qquad (?)$$

$$MH \text{ is } \parallel \text{ to } NO. \qquad (?)$$

$$\therefore\ HOMN \text{ is a } ▱. \qquad (?)$$

$$\therefore\ MN \text{ is equal and } \parallel \text{ to } HO. \qquad (?)$$

In like manner it may be proved that *MN* is equal and ‖ to *OK*.

∴ *HK* is a st. line which is bisected at *O*. (?)

Hence every st. line passing through *O* and terminating in the perimeter is bisected at *O*, and the figure is symmetrical with respect to *O* as a centre. (?)

Q.E.D.

EXERCISES.

712. A circle is symmetrical with respect to its centre, or with respect to any diameter as an axis.

713. A segment of a circle is symmetrical with respect to the perpendicular bisector of its chord as an axis.

714. An isosceles triangle is symmetrical with respect to the perpendicular drawn from the vertex to the base as an axis.

715. An equilateral triangle is symmetrical with respect to a median as an axis.

716. A square is symmetrical with respect to a diagonal as an axis.

717. A parallelogram is symmetrical with respect to the point of intersection of its diagonals as a centre.

718. A regular hexagon is symmetrical with respect to its centre.

719. Two straight lines which are symmetrical with respect to a centre are equal and parallel.

720. If a quadrilateral has two pairs of adjacent sides equal, it is symmetrical with respect to the diagonal joining the vertices of the angles formed by the equal sides as an axis, and the two diagonals are perpendicular to each other.

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